

Hamiltonicity in Split Graphs- a dichotomy

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Abstract. In this paper, we investigate the well-studied Hamiltonian cycle problem, and present an interesting dichotomy result on split graphs. Akiyama et al.[21] have shown that the Hamiltonian cycle problem is NP-complete in planar bipartite graph with maximum degree 3. Using this reduction, we show that the Hamiltonian cycle problem is NP-complete in split graphs. In particular, we show that the problem is NP-complete in $K_{1,5}$ -free split graphs. Further, we present polynomial-time algorithms for Hamiltonian cycle in $K_{1,3}$ -free and $K_{1,4}$ -free split graphs. We believe that the structural results presented in this paper can be used to show similar dichotomy result for Hamiltonian path and other variants of Hamiltonian cycle problem.

1 Introduction

The Hamiltonian cycle (path) problem is a well-known decision problem which asks for the presence of a spanning cycle (path) in a graph. Hamiltonian problems play a significant role in various research areas such as operational research, physics and genetic studies [6,3,8]. This well-known problem has been studied extensively in the literature, and is NP-complete in general graphs. Various sufficient conditions for the existence of Hamiltonian cycle were introduced by Dirac, Ore, Bondy, and A. Ainouche, which were further generalized by A.Kemnitz et al.[2]. A general study on the sufficient conditions were produced by H.J.Broersma and R.J.Gould [9,17,18]. Hamiltonicity has been looked at with respect to various structural parameters, the popular one is toughness. A relation between graph toughness, introduced by Chvatal [22] and Hamiltonicity has been well studied. A detailed survey on Hamiltonicity and toughness is presented in [17,4]. In split graphs, it is proved by D. Kratsch et al. [7] that $\frac{3}{2}$ tough split graphs are Hamiltonian and due to Chvatal's result [22], Hamiltonian graphs are 1-tough. Therefore, the split graphs are Hamiltonian if and only if the toughness is in the range $1.. \frac{3}{2}$.

On algorithmic front, the Hamiltonian cycle problem is NP-complete in chordal [1], chordal bipartite [10], planar [13], and bipartite [21] graphs. Further, Akiyama et al.[21] have shown that the problem is NP-complete in bipartite graph with maximum degree 3. There is a simple reduction for the Hamiltonian cycle problem in bipartite graphs of maximum degree 3 to the Hamiltonian cycle problem in split graphs which we show as part of our dichotomy. Inspite of the hardness of the Hamiltonian problem in various graph classes, nice polynomial-time algorithms have been obtained in interval, circular arc, 2-trees, and distance hereditary graphs [11,20,23,19].

Despite several attempts by researchers, even on special graph classes, we know either a necessary condition or sufficient condition, but not both. Split graphs are a popular subclass of chordal graphs and on which Burkard and Hammer presented a necessary condition [16]. Subsequently, Tan et al. [14] have shown that the necessary condition of [16] is sufficient for some special split graphs. In this paper, we shall revisit Hamiltonicity restricted to split graphs and present a dichotomy result. We show that Hamiltonian cycle is NP-complete in $K_{1,5}$ -free split graphs and polynomial-time solvable in $K_{1,4}$ -free split graphs. It is important to note that a very few NP-complete problems have dichotomy results in the literature [12,15].

We use standard basic graph-theoretic notations. Further, we follow [5]. All the graphs we mention are simple, and unweighted. Graph G has vertex set $V(G)$ and edge set $E(G)$ which we denote using V, E , respectively, once the context is unambiguous. For *minimal vertex separator*, *maximal clique*, and *maximum clique* we use the standard definitions. A graph G is 2-connected if every minimal vertex separators are

of size at least two. Split graphs are $C_4, C_5, 2K_2$ -free graphs and the vertex set of a Split graph can be partitioned into a clique K and an independent set I . For a Split graph with vertex partitions K and I , we assume K to be a maximum clique. For every $v \in K$ we define $N^I(v) = N(v) \cap I$, where $N(v)$ denotes the neighborhood of vertex v . $d^I(v) = |N^I(v)|$ and $\Delta^I = \max\{d^I(v) : v \in K\}$. We define an n -book H as follows; $V(H) = \{u, v, v_1, \dots, v_{n-2}\}$ and $E(H) = \{uv\} \cup \bigcup_{1 \leq i \leq n-2} \{uv_i, vv_i\}$. For a cycle or a path C , we use

\vec{C} to represent an ordering of the vertices of C in one direction (forward direction) and \overleftarrow{C} to represent the ordering in the other direction. $u\vec{C}v$ represents the ordered vertices from u to v in C . For two paths P and Q , $P \cap Q$ denotes $V(P) \cap V(Q)$. For simplicity, we use P to denote the underlying set $V(P)$.

2 Hamiltonian problem in split graphs - polynomial results

In this section we shall present structural results on some special split graphs using which we can find Hamiltonian cycle in such graphs. In particular we explore the structure of $K_{1,3}$ -free split graphs and $K_{1,4}$ -free split graphs.

Lemma 1. [12] *For a claw-free split graph G , if $\Delta^I = 2$, then $|I| \leq 3$.*

Observation 1 *2-connected Split graph G with $\Delta^I = 1$ has a Hamiltonian cycle.*

Lemma 2. *Let G be a $K_{1,3}$ -free split graph. G contains a Hamiltonian cycle if and only if G is 2-connected.*

Proof. Necessity is trivial. For the sufficiency, we consider the following cases.

Case1: $|I| \geq 4$. As per Lemma 1, $\Delta^I = 1$ and by Observation 1, G has a Hamiltonian cycle.

Case2: $|I| \leq 3$. If $\Delta^I = 1$, then by Observation 1, G has a Hamiltonian cycle. When $\Delta^I > 1$, note that either $|I| = 2$ or $|I| = 3$.

(a) $|I| = 2$, i.e., $I = \{s, t\}$. Since $\Delta^I = 2$, there exists a vertex $v \in K$ such that $d^I(v) = 2$ and $N^I(v) = \{s, t\}$. Let $S = N(s)$ and $T = N(t)$. Clearly, $K = S \cup T$. Suppose the set $S \setminus T$ is empty, then the vertices $T \cup \{t\}$ induces a clique, larger in size than K , contradicting the maximality of K . Therefore, the set $S \setminus T$ is non-empty. Similarly, $T \setminus S \neq \emptyset$. It follows that, $|K| \geq 3$; let $x, v, w \in K$ such that $x \in T \setminus S$ and $w \in S \setminus T$. Further, $(w, s, v, t, x, v_1, v_2, \dots, v_k, w)$ is a Hamiltonian cycle in G where $\{v_1, v_2, \dots, v_k\} = K \setminus \{v, w, x\}$.

(b) If $|I| = 3$ and let $I = \{s, t, u\}$. Since G is a 2-connected claw free graph, clearly $|K| \geq 3$. Since $\Delta^I = 2$, there exists $v \in K$ such that $N^I(v) = \{s, t\}$. Further, since G is claw free, for every vertex $w \in K$, $N^I(w) \cap \{s, t\} \neq \emptyset$. Let $S = N(s)$ and $T = N(t)$. Suppose the set $S \setminus T$ is empty, then the vertices $T \cup \{t\}$ induces a clique, larger in size than K , contradicting the maximality of K . Therefore, the set $S \setminus T \neq \emptyset$. Similarly, $T \setminus S \neq \emptyset$. Note that $|N(u)| \geq 2$ as G is 2-connected. If u is adjacent to a vertex $v \in S \cap T$, then $\{v\} \cup N^I(v)$ induces a claw. Therefore, $N(u) \cap (S \cap T) = \emptyset$. It follows that u is adjacent to some vertices in $S \setminus T$ or $T \setminus S$. Suppose $N(u) \cap (S \setminus T) = \emptyset$ and $N(u) \cap (T \setminus S) \neq \emptyset$, then there exists a vertex $z \in T \setminus S$ such that $N^I(z) = \{t, u\}$. Since $S \setminus T \neq \emptyset$, there exists $z' \in S \setminus T$ and $\{z, u, t, z'\}$ induces a claw, a contradiction. Therefore, $N(u) \cap (S - T) \neq \emptyset$ and similarly, $N(u) \cap (T - S) \neq \emptyset$. It follows that there exists a vertex $x \in (S - T)$, $y \in (T - S)$ such that $x, y \in N(u)$. If $|K| = 3$, then (x, s, v, t, y, u, x) is a Hamiltonian cycle in G . Further, if $|K| \geq 3$, then $(x, s, v, t, y, u, x, w_1, \dots, w_l)$ is the desired Hamiltonian cycle where $w_1, \dots, w_l \in K$. This completes the case analysis, and the proof of Lemma 2. \square

$K_{1,4}$ -free split graphs

Now we shall present structural observations in $K_{1,4}$ -free split graphs. The structural results in turn gives a polynomial-time algorithm for the Hamiltonian cycle problem, which is one part of our dichotomy. From Observation 1, it follows that a split graph G with $\Delta^I = 1$ has a Hamiltonian cycle if and only if G is 2-connected. It is easy to see that for $K_{1,4}$ -free split graphs, $\Delta^I \leq 3$, and thus we analyze such a graph G in two variants, $\Delta_G^I = 2$, and $\Delta_G^I = 3$.

Claim A: *For a split graph G with $\Delta^I = 3$, let $v \in K$, $d^I(v) = 3$, and $U = N^I(v)$. If G is $K_{1,4}$ -free, then $N(U) = K$.*

Proof. If not, let $w \in K$ such that $w \notin N(U)$. Clearly, $N^I(v) \cup \{w, v\}$ induces a $K_{1,4}$, a contradiction. \square

Claim B: For a $K_{1,4}$ -free split graph G with $\Delta^I = 3$, let $v \in K$ such that $d^I(v) = 3$, and the split graph $H = G - N^I(v)$. Then, $\Delta_H^I \leq 2$.

Proof. Otherwise, if there exists $x \in K$, $d_H^I(x) = 3$, then $N^I(v) \cup \{x, v\}$ induces a $K_{1,4}$ in G , a contradiction. \square

The next theorem shows a necessary and sufficient condition for the existence of Hamiltonian cycle in split graphs with $\Delta^I = 2$. We define the notion *short cycle* in a $K_{1,4}$ -free split graph G . Consider the subgraph H of G where $V_a = \{u \in I : d(u) = 2\}$, $V_b = N(V_a)$, $V(H) = V_a \cup V_b$ and $E(H) = \{uv : u \in V_a, v \in V_b\}$. Clearly, H is a bipartite subgraph of G . Let C be an induced cycle in H such that $V(K) \setminus V(C) \neq \emptyset$. We refer to C in H as a short cycle in G .

Theorem 1. Let G be a $K_{1,4}$ -free split graph with $\Delta^I = 2$. G has a Hamiltonian cycle if and only if there are no short cycles in G .

Proof. Necessity is trivial as, if there exists a short cycle D , then $c(G - S) > |S|$ where $S = V(D) \cap K$. For sufficiency: if $|I| = |K|$ and G has no short cycles, then clearly, H is a spanning cycle of G . Further, if G has no short cycles with $|K| > |I|$, then note that H is a collection of paths P_1, \dots, P_i , all of them are having end vertices in K . Let $V' = I \setminus V_a$. If $V' = \emptyset$, then it is easy to join the paths using clique edges to get a Hamiltonian cycle of G . Otherwise we partition the vertices in V' into three sets V_2, V_1, V_0 where $V_2 = \{u \in V' : N(u) \cap P_j \neq \emptyset \text{ and } N(u) \cap P_k \neq \emptyset, 1 \leq j \neq k \leq i\}$, $V_1 = \{u \in V' : N(u) \cap P_j \neq \emptyset, 1 \leq j \leq i \text{ and } u \notin V_2\}$, $V_0 = \{u \in V' : N(u) \cap P_j = \emptyset, 1 \leq j \leq i\}$. From the definitions, vertices in V_2 are adjacent to the end vertices of at least two paths, vertices in V_1 are adjacent to the end vertices of exactly one path and that of V_0 are not adjacent to the end vertices of any paths. Now we obtain two graphs H_1 , and H_2 from H and finally, we see that H_2 is a collection of paths containing all the vertices of I , all those paths having end vertices in K . We iteratively add the vertices in V_2 and V_1 into H to obtain H_1 , based on certain preferences till $V_2 = V_1 = \emptyset$. We pick a vertex (from V_2 if $V_2 \neq \emptyset$, otherwise from V_1) and add it to H . If we add a vertex u from V_2 , then we join two arbitrary paths each has its end vertex adjacent to u . Therefore, the addition of a vertex from V_2 reduces the number of paths in H by one. If $u \in V_1$, then u is added to H in such a way that one of its end vertex is an end vertex of a path and the other is not an end vertex of any paths. Clearly, such two vertices are possible due to the fact that $d(u) \geq 3$. Note that in this case, one of the paths in H gets its size increased, still in both cases all the paths have their end vertices in K . We add the vertices in such a way that the vertices in V_2 gets preference over that of V_1 . Also, after every addition of a vertex, we add the new vertex to V' and re-compute the partitions V_2, V_1 and V_0 . Observe that once the set V_2 is empty, and a vertex from V_1 is added, the re-computation may result in a case where $V_2 \neq \emptyset$. Once $V_2 = V_1 = \emptyset$, the graph H_1 obtained is a collection of one or more paths having end vertices in K .

Now we continue the addition by including the vertices of V_0 . A vertex $u \in V_0$ is added in such a way that it forms a P_3 with two of its arbitrary neighbors in K . Evidently, the addition of vertices from V_0 increases the number of paths by one. Note that the addition of vertices from V_0 may result in $V_2 \neq \emptyset$ or $V_1 \neq \emptyset$. Therefore, when we iteratively add the vertices to H_1 , we give first preference to the vertices in V_2 , then to the vertices in V_1 (if $V_2 = \emptyset$) and finally to that in V_0 (if $V_2 = V_1 = \emptyset$). Here also, after each addition of a vertex u , we add u to V' and re-compute the partitions. H_2 is the graph obtained by iteratively adding all the vertices left in I . It is interesting to see that the number of paths in H_2 is at least the number of paths in H_1 . Finally, in H_2 we have a collection of one or more paths with end vertices in K . It is easy to see that the paths could be joined using clique edges to get a Hamiltonian cycle in G . This completes the proof. \square

Having presented a characterization for the Hamiltonian cycle problem in split graphs with $\Delta^I \leq 2$, we shall now present our main result, which is a necessary and sufficient condition for the existence of Hamiltonian cycle in $K_{1,4}$ -free split graphs. Note that for a $K_{1,4}$ -free split graph G , $\Delta^I \leq 3$ and thus the left over case to analyze is when $\Delta^I = 3$. When $\Delta^I = 3$, there exists a vertex $v \in C$ with $d^I(v) = 3$. We obtain $H = G - N^I(v)$, and from Claim B, $\Delta_H^I \leq 2$. Now we shall observe some characteristics of H .

Let G be a 2-connected $K_{1,4}$ -free split graph with $\Delta^I = 3$, $|K| \geq |I| \geq 8$ and $H = G - N^I(v)$ where $v \in K$, $N^I(v) = \{v_1, v_2, v_3\}$. If there are no induced short cycles in G , then by the constructive proof of Theorem 1, in H there exists a collection of vertex disjoint paths \mathbb{C} . Note that each path in \mathbb{C} alternates between an element in K and an element in I , and all the paths are having the end vertices in K . Therefore the paths are having odd number of vertices. Thus, $\mathbb{C} = \mathbb{P}_1 \cup \mathbb{P}_3, \dots, \mathbb{P}_{2i+1}$, where \mathbb{P}_j is the set of maximal paths of size j where for every $P \in \mathbb{P}_j$, there does not exist $P' \in \mathbb{C}$ such that $E(P) \subset E(P')$. A path $P_a \in \mathbb{C}$ is defined on the vertex set $V(P_a) = \{w_1, \dots, w_j, x_1, \dots, x_{j-1}\}$, $E(P_a) = \{w_i x_i : 1 \leq i \leq j-1\} \cup \{w_k x_{k-1} : 2 \leq k \leq j\}$ such that $\{w_1, \dots, w_j\} \subseteq K$, $\{x_1, \dots, x_{j-1}\} \subseteq I$. We denote such a path as $P_a = P(w_1, \dots, w_j; x_1, \dots, x_{j-1})$. We shall now present our structural observations on paths in \mathbb{C} .

Claim 1 *If there exists a path $P_a \in \mathbb{P}_i, i \geq 11$ such that $P_a = P(w_1, \dots, w_j; x_1, \dots, x_{j-1}), j \geq 6$, then there exists $v_1 \in N^I(v)$ such that $v_1 w_i \in E(G), 2 \leq i \leq j-1$.*

Proof. First we show that for every non-consecutive $2 \leq i, k \leq j-1$, for the pair of vertices $w_i, w_k, v_1 w_i \in E(G)$ and $v_1 w_k \in E(G)$. Suppose, if exactly one of w_i, w_k is adjacent to v_1 , say $v_1 w_i \in E(G)$, then $N^I(w_i) \cup \{w_i, w_k\}$ has an induced $K_{1,4}$. If $v_1 w_i \notin E(G)$ and $v_1 w_k \notin E(G)$, then by Claim A, there exists an adjacency for w_i, w_k in v_2, v_3 . Further, if either v_2 or v_3 is adjacent to both w_i, w_k , then $v_1 = v_2$, or $v_1 = v_3$, and the claim is true. Therefore, we shall assume without loss of generality, $v_2 w_i \in E(G)$ and $v_2 w_k \notin E(G)$. This implies that $N^I(w_i) \cup \{w_i, w_k\}$ has an induced $K_{1,4}$, a contradiction. Since the above observation is true for all such pair of vertices in $W = \{w_i\}, 2 \leq i \leq j-1$ and $|W| \geq 4$, it follows that there exists $v_1 \in N^I(v)$ such that $v_1 w_i \in E(G), 2 \leq i \leq j-1$. \square

Claim 2 $\mathbb{P}_i = \emptyset, i \geq 13$.

Proof. Assume for a contradiction that there exists $P_a \in \mathbb{P}_i, i \geq 13$. Let $P_a = P(w_1, \dots, w_j; x_1, \dots, x_{j-1}), j \geq 7$. From Claim 1 there exists $v_1 \in N_G^I(v)$ such that $v_1 w_k \in E(G), 2 \leq k \leq j-1$. Since the clique is maximum in G , there exists $s \in K$ such that $v_1 s \notin E(G)$. Further, there exists at least three vertices in x_1, \dots, x_6 adjacent to s , otherwise, for some $2 \leq r \leq j-1$, $N_G^I(w_r) \cup \{w_r, s\}$ induces a $K_{1,4}$. Finally, from Claim A, either $v_2 s \in E(G)$ or $v_3 s \in E(G)$. It follows that $\{s\} \cup N_G^I(s)$ has an induced $K_{1,4}$, a contradiction. \square

Claim 3 *Let $P_a = P(w_1, \dots, w_i; x_1, \dots, x_{i-1}), i \geq 3$, and $P_b = P(s_1, \dots, s_j; t_1, \dots, t_{j-1}), j \geq 3$ be arbitrary paths in \mathbb{C} . There exists $v_1 \in N^I(v)$ such that $\forall 2 \leq l \leq i-1, v_1 w_l \in E(G)$, and $\forall 2 \leq m \leq j-1, v_1 s_m \in E(G)$.*

Proof. We shall consider every pair of vertices w_l, s_m and show that $v_1 w_l, v_1 s_m \in E(G)$. Suppose, if exactly one of w_l, s_m is adjacent to v_1 , say $w_l v_1 \in E(G)$, then $N^I(w_l) \cup \{w_l, s_m\}$ has an induced $K_{1,4}$. If $v_1 w_l \notin E(G)$ and $v_1 s_m \notin E(G)$, then by Claim A, there exists an adjacency for w_l, s_m in v_2, v_3 . Further, if either v_2 or v_3 is adjacent to both w_l, s_m , then $v_1 = v_2$, or $v_1 = v_3$, and the claim is true. Therefore, we shall assume without loss of generality, $v_2 w_l \in E(G)$ and $v_2 s_m \notin E(G)$. This implies that $N^I(w_l) \cup \{w_l, s_m\}$ has an induced $K_{1,4}$, a contradiction. \square

Corollary 1. *Let $P_a = P(w_1, \dots, w_i; x_1, \dots, x_{i-1}), i \geq 3$, $P_b = P(s_1, \dots, s_j; t_1, \dots, t_{j-1}), j \geq 3$ and $P_c = P(y_1, \dots, y_k; z_1, \dots, z_{k-1}), k \geq 3$ be arbitrary paths in \mathbb{C} . There exists $v_1 \in N^I(v)$ such that $\forall 2 \leq l \leq i-1, v_1 w_l \in E(G)$, $\forall 2 \leq m \leq j-1, v_1 s_m \in E(G)$, and $\forall 2 \leq n \leq k-1, v_1 y_n \in E(G)$.*

Proof. From Claim 3, $\forall 2 \leq l \leq i-1, v_1 w_l \in E(G)$, and $\forall 2 \leq m \leq j-1, v_1 s_m \in E(G)$. Similarly, $\forall 2 \leq l \leq i-1, v_1 w_l \in E(G)$, and $\forall 2 \leq n \leq k-1, v_1 y_n \in E(G)$. Thus the corollary follows from Claim 3. \square

Claim 4 *If there exists $P_a \in \mathbb{P}_{11}$, then $\mathbb{P}_j = \emptyset, j \neq 11, j \geq 5$.*

Proof. Assume for a contradiction that there exists such a path $P_b \in \mathbb{P}_j, j \geq 5$. Let $P_a = (w_1, \dots, w_6; x_1, \dots, x_5)$ and $P_b = (s_1, \dots, s_r; t_1, \dots, t_{r-1}), r \geq 3$. From Claim 1, there exists a vertex $v_1 \in N^I(v)$, such that $v_1 w_i \in E(G), 2 \leq i \leq 5$ and from Claim 3, $v_1 s_j \in E(G), 2 \leq j \leq r-1$. Now we claim $v_1 w_1 \in E(G)$. Otherwise, by Claim A, $v_2 w_1$ or $v_3 w_1$ is in $E(G)$. Observe that either $w_1 x_2 \in E(G)$ or $w_1 x_3 \in E(G)$, otherwise

$N^I(w_3) \cup \{w_3, w_1\}$ induces a $K_{1,4}$. Similarly, either $w_1x_4 \in E(G)$ or $w_1x_5 \in E(G)$. Now $\{w_1\} \cup N^I(w_1)$ induces a $K_{1,4}$. Using similar argument, we establish $v_1w_6 \in E(G)$. Since the clique is maximal, there exists a vertex $w' \in K$ such that $v_1w' \notin E(G)$. We see the following cases. *Case 1:* $w' = s_1$. By Claim A, $s_1v_2 \in E(G)$ or $s_1v_3 \in E(G)$. Further, $s_1x_2 \in E(G)$, otherwise $N^I(w_2) \cup \{w_2, s_1\}$ induces a $K_{1,4}$. Similarly, $s_1x_4 \in E(G)$. Now $\{s_1\} \cup N^I(s_1)$ induces a $K_{1,4}$. Similarly, we could establish a contradiction if $w' = s_r$. *Case 2:* $w' \notin P_i \cup P_j$. By Claim A, $w'v_2 \in E(G)$ or $w'v_3 \in E(G)$. Also due to the similar reasoning for s_1 , $w'x_2, w'x_4 \in E(G)$. Now, either $t_1w' \in E(G)$ or $t_2w' \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, w'\}$ induces a $K_{1,4}$. Finally, $\{w'\} \cup N^I(w')$ induces a $K_{1,4}$, a contradiction. Therefore P_b does not exist. This completes the case analysis and the proof. \square

Claim 5 *If there exists $P_a \in \mathbb{P}_{11}$, then G has a Hamiltonian cycle.*

Proof. Let $P_a = (w_1, \dots, w_6; x_1, \dots, x_5)$. From Claim 1, there exists a vertex say $v_1 \in N_G^I(v)$, such that $v_1w_i \in E(G)$, $2 \leq i \leq 5$. From the proof of the previous claim, $v_1w_1, v_1w_6 \in E(G)$. Since the clique is maximal, there exists $w' \in K$, such that $w'v_1 \notin E(G)$. By Claim A, $w'v_2 \in E(G)$ or $w'v_3 \in E(G)$. Without loss of generality, let $w'v_2 \in E(G)$. We claim $w'x_2 \in E(G)$ and $w'x_4 \in E(G)$, otherwise $N^I(w_2) \cup \{w_2, w'\}$ or $N^I(w_4) \cup \{w_4, w'\}$ induces a $K_{1,4}$, respectively. One among v_2, x_2, x_4 is adjacent to w_1 , otherwise $N^I(w') \cup \{w_1, w'\}$ induces a $K_{1,4}$. Similar argument holds good with respect to the vertex w_6 . Note that for every $t \in \{v, w', w_1, \dots, w_6\}$, $d^I(t) = 3$ and there exists a vertex $w'' \in K$, $w'' \neq t$, where $w''v_3 \in E(G)$. Now we claim $w''v_1 \in E(G)$. If not, for some $1 \leq j \leq 6$, $N^I(w_j) \cup \{w_j, w''\}$ induces a $K_{1,4}$. Finally $(w_1 \overrightarrow{P_a} w_6, v_1, w'', v_3, v, v_2, w')$ is a (w_1, w') path containing all the vertices of $P_a \cup \{v, w', w''\} \cup N^I(v)$, which could be easily extended to a Hamiltonian cycle in G using clique edges to join other vertex disjoint paths. \square

Claim 6 *If there exists $P_a \in \mathbb{P}_9$, then $\mathbb{P}_j = \emptyset, j \neq 9, j \geq 5$.*

Proof. Assume for a contradiction that there exists such a path $P_b \in \mathbb{P}_j, j \geq 5$. Let $P_a = (w_1, \dots, w_5; x_1, \dots, x_4)$ and $P_b = (s_1, \dots, s_r; t_1, \dots, t_{r-1})$, $r \geq 3$. From Claim 3, there exists $v_1 \in N_G^I(v)$ such that $v_1w_2, v_1w_3, v_1w_4, v_1s_i \in E(G)$, $2 \leq i \leq r-1$. Now we claim that $w_1v_1 \in E(G)$. Suppose not, by Claim A either $w_1v_2 \in E(G)$ or $w_1v_3 \in E(G)$. Observe that $w_1x_3 \in E(G)$, otherwise $N^I(w_3) \cup \{w_3, w_1\}$ induces a $K_{1,4}$. Similarly, either $w_1t_1 \in E(G)$ or $w_1, t_2 \in E(G)$ otherwise $N^I(s_2) \cup \{s_2, w_1\}$ induces a $K_{1,4}$. It follows that $\{w_1\} \cup N^I(w_1)$ induces a $K_{1,4}$. This contradicts the assumption that $w_1v_1 \notin E(G)$, and thus $w_1v_1 \in E(G)$. Similar argument holds good with other end vertices of paths P_a, P_b , and hence, $w_5v_1, s_1v_1, s_rv_1 \in E(G)$. Finally, we claim that $w'v_1 \in E(G)$ for every vertex $w' \in K$, where $w' \notin \{w_1, \dots, w_5, s_1, \dots, s_r\}$. If not, let $w'v_1 \notin E(G)$. By Claim A, either v_2w' or v_3w' is in $E(G)$. Further, there exists at least two vertices in x_1, \dots, x_5 adjacent to w' , otherwise, for some $2 \leq i \leq 4$, $N_G^I(w_i) \cup \{w_i, w'\}$ induces a $K_{1,4}$. Now either $w't_1 \in E(G)$ or $w't_2 \in E(G)$, if not, $N^I(s_2) \cup \{s_2, w'\}$ induces a $K_{1,4}$. Therefore, $\{w'\} \cup N^I(w')$ induces a $K_{1,4}$, a contradiction. From the above, we conclude that $\{x\} \cup K$ is a larger clique, which finally contradicts the existence of P_b . Thus if $P_a \in \mathbb{C}$, then there does not exist such a path P_b . This completes the proof. \square

In the following claims to show the existence of Hamiltonian cycle, we shall do a constructive approach in which we produce a (u, v) -path where $u, v \in K$. The path is obtained by joining some paths in \mathbb{C} using the vertices in $N^I(v)$. Therefore such a *desired path* is sufficient to show that G has a Hamiltonian cycle, which is in turn obtained by joining all such vertex disjoint paths using clique edges.

Claim 7 *If there exists $P_a \in \mathbb{P}_9$ and G has no short cycles, then G has a Hamiltonian cycle.*

Proof. Let $P_a = (w_1, \dots, w_5; x_1, \dots, x_4)$. Since $|I| \geq 8$ and by Claim 6, there exists at least one more path $P_b \in \mathbb{P}_3$ such that $P_b = (s_1, s_2; t_1)$. There exists a vertex in $N^I(v) = \{v_1, v_2, v_3\}$ adjacent to w_2 , say $v_1w_2 \in E(G)$. Note that $v_1w_4 \in E(G)$, otherwise $\{w_2, w_4\} \cup N^I(w_2)$ induces a $K_{1,4}$. Since the clique is maximal, there exists a non-adjacency for v_1 in K , and based on the non-adjacency, we see the following cases.

Case 1: $v_1w_1 \notin E(G)$ or $v_1w_5 \notin E(G)$. Without loss of generality, we shall assume $v_1w_1 \notin E(G)$. Note that one of v_2, v_3 is adjacent to w_1 , say $v_2w_1 \in E(G)$. Note that $w_1x_3 \in E(G)$ or $w_1x_4 \in E(G)$, if not $N^I(w_4) \cup \{w_4, w_1\}$ has an induced $K_{1,4}$. Note that $w_3v_1 \in E(G)$ or $w_3v_2 \in E(G)$ or $w_3v_3 \in E(G)$. Observe that if $w_1x_4 \in E(G)$, then $w_3v_1 \notin E(G)$ and $w_3v_3 \notin E(G)$. Therefore there exists four possibilities as follows.

Case 1.1: $w_1x_3, w_3v_1 \in E(G)$. We shall see the adjacency of the vertices s_1 and s_2 with respect to $\{v_1, v_2, v_3\}$. We observe that either $v_1s_i, v_2s_i \in E(G)$ or $v_1s_i, x_1s_i \in E(G)$ or $v_1s_i, x_3s_i \in E(G)$, $i \in \{1, 2\}$. Clearly, $v_1s_i \in E(G)$. If $w_5v_1 \in E(G)$, then note that $d^I(w_j) = d^I(s_i) = 3$, $1 \leq j \leq 5, i \in \{1, 2\}$, and therefore there exists a vertex $w' \in K$, $w' \notin \{w_1, \dots, w_5, s_1, s_2\}$ such that $w'v_3 \in E(G)$. Finally, the path $P_1 = (w', v_3, v, v_2, w_1 \xrightarrow{P_a} w_5, v_1, s_1, t_1, s_2)$ is a desired path. If w_5 is adjacent to v_2, x_2 , then note that $v_1s_i, v_2, s_i \in E(G)$. Then $(w', v_3, v, v_2, s_1, t_1, s_2, v_1, w_4, x_4, w_5, x_2, w_3, x_3, w_1, x_1, w_1)$ is a desired path. Since the path has its end points in K , we get a Hamiltonian cycle.

Case 1.2: $w_1x_3, w_3v_2 \in E(G)$. Similar to the previous case, either $v_1s_i, v_2s_i \in E(G)$ or $v_1s_i, x_3s_i \in E(G)$, $i \in \{1, 2\}$ and thus $v_1s_i \in E(G)$. Further, similar to previous case, there exists $w' \in K$ such that $w'v_3 \in E(G)$. Now if $w_5v_1 \in E(G)$, then the path P_1 as obtained in the previous case is a desired path. If $w_5v_1 \notin E(G)$, then either $w_5v_2, w_5x_1 \in E(G)$ or $w_5v_2, w_5x_2 \in E(G)$. Thus $P_2 = (w', v_3, v, v_2, w_1, x_1, w_5 \xleftarrow{P_a} w_2, v_1, s_1, t_1, s_2)$ or $(w', v_3, v, v_2, w_3, x_2, w_5, x_4, w_4, x_3, w_1, x_1, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

Case 1.3: $w_1x_4, w_3v_2 \in E(G)$. Note that $v_1s_i, v_2s_i \in E(G)$, $i \in \{1, 2\}$. Further, similar to previous case, there exists $w' \in K$ such that $w'v_3 \in E(G)$. Now if $w_5v_1 \in E(G)$, then the path P_1 as obtained in the Case 1.1 is a desired path. If $w_5v_1 \notin E(G)$, then either $w_5v_2, w_5x_1 \in E(G)$ or $w_5v_2, w_5x_2 \in E(G)$. Thus the path P_2 as obtained in previous case or $(w', v_3, v, v_2, w_5, x_4, w_1 \xrightarrow{P_a} w_4, v_1, s_1, t_1, s_2)$ is a desired path.

Case 1.4: $w_1x_3, w_3v_3 \in E(G)$. Note that $v_1s_i, x_3s_i \in E(G)$, $i \in \{1, 2\}$. If w_5 is adjacent to v_1, x_3 , then consider $S = \{v_2, v_3, x_1, x_2\}$. Further, for every $x' \in S$, $d^I(x') = 2$, and $S \cup N(S)$ has a short cycle. Since G has no short cycles, there exists a vertex $w' \in K$ such that w' is adjacent to some vertices in S . In this case one of the following is a desired path.

$(w', v_2, v, v_3, w_3, x_2, w_2, x_1, w_1, x_3, w_4, x_4, w_5, v_1, s_1, t_1, s_2)$

$(w', v_3, v, v_2, w_1 \xrightarrow{P_a} w_5, v_1, s_1, t_1, s_2)$

$(w', x_1 \xrightarrow{P_a} w_3, v_3, v, v_2, w_1, x_3 \xrightarrow{P_a} w_5, v_1, s_1, t_1, s_2)$

$(w', x_2, w_2, x_1, w_1, v_2, v, v_3, w_3 \xrightarrow{P_a} w_5, v_1, s_1, t_1, s_2)$

Now we see the case in which w_5 is adjacent to v_2, x_1 or v_3, x_3 . Note that

$(s_2, t_1, s_1, v_1, w_4, x_4, w_5, v_2, v, v_3, w_3, x_2, w_2, x_1, w_1, x_3, s_2)$ or

$(s_2, t_1, s_1, v_1, w_4, x_4, w_5, v_3, v, v_2, w_1 \xrightarrow{P_a} x_3, s_2)$ are possible Hamiltonian cycles.

Case 2: $v_1w_3 \notin E(G)$. In this case we shall assume that $v_1w_i \in E(G)$, $i \in \{1, 2, 4, 5\}$. Note that $w_3v_2 \in E(G)$ or $w_3v_3 \in E(G)$, say $w_3v_2 \in E(G)$. Clearly, $s_iv_1, s_iv_2 \in E(G)$ or $s_iv_1, s_ix_2 \in E(G)$ or $s_iv_1, s_ix_3 \in E(G)$, $i \in \{1, 2\}$. Similarly, w_1, w_5 are adjacent to one of v, x_2, x_3 . Therefore there exists $w' \in K$ such that $w'v_3 \in E(G)$. We see the following cases depending on the adjacency of s_i, w_1, w_5 with $N^I(w_3)$.

Case 2.1: $w_1v_2 \in E(G)$ or $w_5v_2 \in E(G)$, say $w_1v_2 \in E(G)$.

$(w', v_3, v, v_2, w_1 \xrightarrow{P_a} w_5, v_1, \vec{P}_b)$ is a desired path.

Case 2.2: $w_1x_3 \in E(G)$ or $w_5x_2 \notin E(G)$. $(w', v_3, v, v_2, w_3, x_2 \xleftarrow{P_a} w_1, x_3 \xrightarrow{P_a} w_5, v_1, \vec{P}_b)$ or

$(w', v_3, v, v_2, w_3, x_3 \xrightarrow{P_a} w_5, x_2 \xleftarrow{P_a} w_1, v_1, \vec{P}_b)$ is a desired path.

Case 2.3: $w_1v_2, w_5v_3 \notin E(G)$. If $s_iv_2 \in E(G)$, say $s_1v_2 \in E(G)$, then $(w', v_3, v, v_2, s_1, t_1, s_2, v_1, w_1 \xrightarrow{P_a} w_5)$ is a desired path. If $s_ix_2 \in E(G)$, say $s_1x_2 \in E(G)$, then $(w', v_3, v, v_2, w_3 \xrightarrow{P_a} w_5, v_1, w_1 \xrightarrow{P_a} x_2, s_1, t_1, s_2)$ is a desired path. If $s_ix_3 \in E(G)$, say $s_1x_3 \in E(G)$, then $(w', v_3, v, v_2, w_3 \xleftarrow{P_a} w_1, v_1, w_5 \xleftarrow{P_a} x_3, s_1, t_1, s_2)$ is a desired path.

Case 3: $v_1w' \notin E(G)$, $w' \notin P_a$. In this case we shall assume that $v_1w_i \in E(G)$, $1 \leq i \leq 5$. Note that in this case $w' \notin P_b$. Suppose $w' \in P_b$, say $v_1s_1 \notin E(G)$, then s_1 is adjacent to at least two vertices in x_1, \dots, x_4 . Further, either $s_1v_2 \in E(G)$ or $s_1v_3 \in E(G)$. Thus $\{s_1\} \cup N^I(s_1)$ induces a $K_{1,4}$, a contradiction. Therefore $w' \notin P_b$. It follows that w' is an end vertex of some path $P_c \in \mathbb{P}_3 \cup \mathbb{P}_1$. Note that $w'v_2 \in E(G)$ or $w'v_3 \in E(G)$, say $w'v_2 \in E(G)$. If v_3 is adjacent to P_a , then $(\vec{P}_b, v_1, \vec{P}_a, v_3, v, v_2, w' \xrightarrow{P_c})$ is a desired path.

If v_3 is adjacent to $P_d \neq P_a$, then $(\vec{P}_b, v_1, \vec{P}_a, \vec{P}_d, v_3, v, v_2, w' \vec{P}_c)$ is a desired path. In all of the above cases, using the desired paths, we can get a Hamiltonian cycle as the end points are in K .

This completes the case analysis and a proof. \square

Claim 8 $|\mathbb{P}_7| \leq 2$. Further, if $|\mathbb{P}_7| = 2$, then $\mathbb{P}_5 = \emptyset$.

Proof. We first show that $|\mathbb{P}_7| \leq 2$. Suppose that there exists $P_a, P_b, P_c \in \mathbb{P}_7$ such that $P_a = (w_1, \dots, w_4; x_1, \dots, x_3)$, $P_b = (s_1, \dots, s_4; t_1, \dots, t_3)$, and $P_c = (q_1, \dots, q_4; r_1, \dots, r_3)$. By Claim 3, there exists a vertex $v_1 \in N^I(v)$ such that $v_1 w_j, v_1 s_j, v_1 q_j \in E(G), j \in \{2, 3\}$. Since the clique is maximal there exists $w' \in K$ such that $w' v_1 \notin E(G)$. It follows that $w' v_2 \in E(G)$ or $w' v_3 \in E(G)$. We now claim that $w' x_2, w' t_2, w' r_2 \in E(G)$. Suppose $w' x_2 \notin E(G)$, then $N^I(w_2) \cup \{w_2, w'\}$ or $N^I(w_3) \cup \{w_3, w'\}$ induces a $K_{1,4}$. Similar arguments can be given for other edges. Now $\{w'\} \cup N^I(w')$ induces a $K_{1,4}$, a contradiction to the existence of three such paths P_a, P_b, P_c . To prove the second half, let $P_a, P_b \in \mathbb{P}_7$. For a contradiction, assume that $P_d \in \mathbb{P}_5$ such that $P_d = (y_1, \dots, y_3; z_1, z_2)$. From Claim 3, there exists $v_1 \in N^I(v)$ such that $v_1 w_i, v_1 s_i, v_1 y_2 \in E(G), i \in \{2, 3\}$. Now we claim that $v_1 w_i, v_1 s_i, v_1 y_j \in E(G), i \in \{1, 4\}, j \in \{1, 3\}$. Suppose $v_1 w_1 \notin E(G)$, then by Claim A, either $v_2 w_1 \in E(G)$ or $v_3 w_1 \in E(G)$. Note that $w_1 t_2 \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, w_1\}$ or $N^I(s_3) \cup \{s_3, w_1\}$ induces a $K_{1,4}$. Further, $w_1 z_1 \in E(G)$ or $w_1 z_2 \in E(G)$, otherwise $N^I(y_2) \cup \{y_2, w_1\}$ induces a $K_{1,4}$. It follows that $\{w_1\} \cup N^I(w_1)$ induces a $K_{1,4}$, a contradiction to the assumption that $v_1 w_1 \notin E(G)$. Similar arguments hold good for the other edges. Finally, for any arbitrary vertex $w' \in K$ where $w' \notin \{w_1, \dots, w_4, s_1, \dots, s_4, y_1, \dots, y_3\}$, we claim $v_1 w' \in E(G)$. If not, then by Claim A, either $v_2 w' \in E(G)$ or $v_3 w' \in E(G)$. Further, similar to the previous argument for the vertex w_1 , the same arguments hold good for w' , i.e., $w' x_2, w' t_2 \in E(G)$ and either $w' z_1 \in E(G)$ or $w' z_2 \in E(G)$. Now, $\{w'\} \cup N^I(w')$ induces a $K_{1,4}$, a contradiction. Thus $v_1 w' \in E(G)$. Therefore, we conclude that $\{v_1\} \cup K$ is a clique of larger size, which is the final contradiction to the existence of P_d . This completes the proof. \square

Claim 9 If $|\mathbb{P}_7| = 1$, then $|\mathbb{P}_5| \leq 1$.

Proof. Let $P_a \in \mathbb{P}_7$, where $P_a = (w_1, \dots, w_4; x_1, \dots, x_3)$. Assume for a contradiction that there exists paths $P_b, P_c \in \mathbb{P}_5$ such that $P_b = (s_1, s_2, s_3; t_1, t_2)$, and $P_c = (q_1, q_2, q_3; r_1, r_2)$. From Claim 3, there exists $v_1 \in N^I(v)$ such that $v_1 w_2, v_1 w_3, v_1 s_2, v_1 q_2 \in E(G)$. Similar to the proof of previous claim we could argue that $v_1 w_1, v_1 w_4, v_1 s_i, v_1 q_i \in E(G), i \in \{1, 3\}$. Further, for any arbitrary vertex $w' \in K$ such that $w' \notin \{w_1, \dots, w_4, s_1, s_2, s_3, q_1, q_2, q_3\}$, we claim $v_1 w' \in E(G)$. Suppose not, then by Claim A, either $v_2 w' \in E(G)$ or $v_3 w' \in E(G)$. Further, similar to the proof of previous claim, we could argue that $w' x_2 \in E(G)$, one of $w' t_1$ or $w' t_2$ is in $E(G)$ and one of $w' r_1$ or $w' r_2$ is in $E(G)$. Now, $\{w'\} \cup N^I(w')$ induces a $K_{1,4}$, which is a contradiction to the assumption that $v_1 w' \notin E(G)$. Therefore, we conclude that $\{v_1\} \cup K$ is a clique of larger size, which is the final contradiction to the existence of two such paths P_b, P_c . This completes the proof. \square

Claim 10 If there exists $P_a, P_b \in \mathbb{P}_7$, then G has a Hamiltonian cycle.

Proof. Let $P_a, P_b \in \mathbb{P}_7$ such that $P_a = (w_1, \dots, w_4; x_1, \dots, x_3)$, $P_b = (s_1, \dots, s_4; t_1, \dots, t_3)$. Similar to the arguments in the proof of Claim 8, there exists $v_1 \in N^I(v)$ such that $v_1 w_i, v_1 s_i \in E(G), 1 \leq i \leq 4$. Since K is a maximal clique, there exists $w' \in K$ such that $w' v_1 \notin E(G)$. From Claim A, either $w' v_2 \in E(G)$ or $w' v_3 \in E(G)$. Without loss of generality, let $w' v_3 \in E(G)$. Note that $w' x_2 \in E(G)$, otherwise, either $N^I(w_2) \cup \{w_2, w'\}$ induces a $K_{1,4}$ or $N^I(w_3) \cup \{w_3, w'\}$ induces a $K_{1,4}$. Similarly, $w' t_2 \in E(G)$. Note that the vertices $w_i, s_i, i \in \{1, 4\}$ is adjacent to one of the vertices in $\{v_3, t_2, x_2\}$, if not, say $w_1 v_3, w_1 t_2, w_1 x_2 \notin E(G)$, then $N^I(w') \cup \{w', w_1\}$ induces a $K_{1,4}$. Similar arguments hold for w_2, s_1, s_2 . It follows that for every $s' \in S = \{w_1, \dots, w_4, s_1, \dots, s_4\}$, $d^I(s') = 3$. Since G is two connected, there exists $w'' \in K \setminus S$ such that $w'' v_2 \in E(G)$. Observe that $(w'', v_2, v, v_3, w', w_1 \vec{P}_a w_4, x, s_1 \vec{P}_b s_4)$ is a path containing $N^I(v)$ which could be easily extended to a Hamiltonian cycle in G . \square

Claim 11 *If there exists $P_a \in \mathbb{P}_7, P_b \in \mathbb{P}_5$ and G has no short cycle, then G has a Hamiltonian cycle.*

Proof. Let $P_a \in \mathbb{P}_7, P_b \in \mathbb{P}_5$, such that $P_a = (w_1, \dots, w_4; x_1, \dots, x_3), P_b = (s_1, \dots, s_3; t_1, \dots, t_2)$. From Claim 3, there exists $v_1 \in N^I(v)$ such that $v_1 w_i, v_1 s_2 \in E(G), i \in \{2, 3\}$. Now we claim that $v_1 w_1 \in E(G)$. If not, observe that either $v_2 w_1 \in E(G)$ or $v_3 w_1 \in E(G)$. Also note that $w_1 x_2 \in E(G)$ or $w_1 x_3 \in E(G)$, otherwise $N^I(w_3) \cup \{w_3, w_1\}$ induces a $K_{1,4}$. Further, $w_1 t_1 \in E(G)$ or $w_1 t_2 \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, w_1\}$ induces a $K_{1,4}$. Clearly, $\{w_1\} \cup N^I(w_1)$ induces a $K_{1,4}$, contradicting $v_1 w_1 \notin E(G)$. Similarly, $x w_4 \in E(G)$. Since the clique is maximal, there exists $w' \in K$ such that $v_1 w' \notin E(G)$. We see the following cases.

Case 1: $w' \notin \{s_1, s_3\}$, thus $v_1 s_1, v_1 s_3 \in E(G)$. From the previous claims, it is easy to see that w' is an end vertex of a path P_c in $\mathbb{P}_3 \cup \mathbb{P}_1$. From Claim A, there exists $v_3 \in N^I(v)$ such that $v_3 w' \in E(G)$. Now we claim $w' x_2 \in E(G)$, otherwise, either $N^I(w_2) \cup \{w_2, w'\}$ induces a $K_{1,4}$ or $N^I(w_3) \cup \{w_3, w'\}$ induces a $K_{1,4}$. Also observe that either $w' t_1$ or $w' t_2$ is in $E(G)$, otherwise $N^I(s_2) \cup \{s_2, w'\}$ induces a $K_{1,4}$. Without loss of generality, let $w' t_2 \in E(G)$. Now note that all the vertices in $\{w_1, w_4, s_1\}$ has an adjacency in $\{t_2, x_2, v_3\}$. Clearly, $d^I(w_j) = d^I(s_1) = d^I(s_2) = 3, 1 \leq j \leq 4$ and since the graph is two connected, $v_2 s_3 \in E(G)$ or $v_2 w'' \in E(G)$ where w'' is the end vertex of a path P_d in $\mathbb{P}_3 \cup \mathbb{P}_1$. Here we obtain $(\vec{P}_c w', v_3, v, v_2, s_3 \vec{P}_b s_1, v_1, w_1 \vec{P}_a w_4)$ or $(\vec{P}_d w'', v_2, v, v_3, w' \vec{P}_c, w_1 \vec{P}_a w_4, v_1, s_1 \vec{P}_b s_3)$ as a desired path.

Case 2: $w' \in \{s_1, s_3\}$. Without loss of generality, let $w' = s_1$, i.e., $v_1 s_1 \notin E(G)$. From Claim A, there exists $v_3 \in N^I(v)$ such that $v_3 s_1 \in E(G)$. Also note that $s_1 x_2 \in E(G)$, otherwise either $N^I(w_2) \cup \{w_2, s_1\}$ induces a $K_{1,4}$ or $N^I(w_3) \cup \{w_3, s_1\}$ induces a $K_{1,4}$. Now we claim that w_1 and w_4 are adjacent to one of the vertices in $S = \{v_3, t_1, x_2\}$. Suppose $w_1 v_3, w_1 t_1, w_1 x_2 \notin E(G)$, then $N^I(s_1) \cup \{s_1, w_1\}$ induces a $K_{1,4}$. Similar arguments hold for w_4 . It follows that $d^I(w_j) = d^I(s_k) = 3, 1 \leq j \leq 4, k \in \{1, 2\}$. Since G is 2-connected, there exists $w^* \in K$ such that $w^* v_2 \in E(G)$. We see the following sub cases based on the possibility of w^* .

Case 2.1: $w^* = s_3$. i.e., $v_2 s_3 \in E(G)$. In this sub case we claim that there exists a vertex $w'' \neq v \in K$ such that $w'' \notin P_a \cup P_b$ and $w'' x' \in E(G)$ where $x' \in \{v_2, v_3, t_1, t_2\}$. Suppose such a w'' does not exist, then observe that, in the set $S = \{t_1, t_2, y, z\}$, $d(t_1) = d(t_2) = d(y) = d(z) = 2$, and $S \cup N(S)$ has a short cycle, a contradiction. Note that, w'' is an end vertex of a path P_d in $\mathbb{P}_3 \cup \mathbb{P}_1$. Now depending on the adjacency of w'' , we obtain the following paths.

If $w'' v_2 \in E(G)$, then we obtain $(\vec{P}_d w'', v_2, s_3 \vec{P}_b s_1, v_3, v, v_1, w_1 \vec{P}_a w_4)$ as a desired path.

If $w'' v_3 \in E(G)$, then we obtain $(\vec{P}_d w'', v_3, s_1 \vec{P}_b s_3, v_2, v, v_1, w_1 \vec{P}_a w_4)$ as a desired path.

If $w'' t_1 \in E(G)$, then we obtain $(\vec{P}_d w'', t_1, s_1, v_3, v, v_2, s_3 \vec{P}_b s_2, v_1, w_1 \vec{P}_a w_4)$ as a desired path.

If $w'' t_2 \in E(G)$, then we obtain $(\vec{P}_d w'', t_2, s_3, v_2, v, v_3, s_1 \vec{P}_b s_2, v_1, w_1 \vec{P}_a w_4)$ as a desired path.

Case 2.2: $w^* \neq s_3$. Note that w^* is an end vertex of a path P_e in $\mathbb{P}_3 \cup \mathbb{P}_1$. We see the following sub cases to complete our argument.

Case 2.2.1: $v_1 s_3 \in E(G)$. Here we obtain $(\vec{P}_e w^*, v_2, v, v_3, s_1 \vec{P}_b s_3, v_1, w_1 \vec{P}_a w_4)$ as a desired path.

Case 2.2.2: $v_1 s_3 \notin E(G)$. Clearly, from Claim A either $s_3 v_2 \in E(G)$ or $s_3 v_3 \in E(G)$. We now claim that $s_3 x_2 \in E(G)$. Otherwise either $N^I(w_2) \cup \{w_2, s_3\}$ induces a $K_{1,4}$ or $N^I(w_3) \cup \{w_3, s_3\}$ induces a $K_{1,4}$.

Here we obtain $(\vec{P}_e w^*, v_2, v, v_3, s_1 \vec{P}_b s_3, x_2 \vec{P}_a w_4, v_1, w_2, x_1, w_1)$ as a desired path.

This completes the case analysis and the proof. \square

Claim 12 *If there exists $P_a \in \mathbb{P}_7, P_b, P_c \in \mathbb{P}_3$ and $\mathbb{P}_5 = \emptyset$ and G has no short cycles, then G has a Hamiltonian cycle.*

Proof. Let $P_a = (w_1, \dots, w_4; x_1, \dots, x_3), P_b = (s_1, s_2; t_1)$, and $P_c = (q_1, q_2; r_1)$. From Claim A, the vertices w_2, w_3 are adjacent to at least one of the vertices in $N^I(v)$. Depending on this adjacency, we see the following two cases.

Case 1: There exists $v_1 \in N^I(v)$ such that $v_1 w_2, v_1 w_3 \in E(G)$.

We first claim that the vertices s_1, s_2, q_1, q_2 are adjacent to at least one of v_1, x_2 . Suppose that $v_1 s_1, x_2 s_1 \notin E(G)$. Note that $v_2 s_1 \in E(G)$ or $v_3 s_1 \in E(G)$. If $s_1 x_1 \notin E(G)$, then $N^I(w_2) \cup \{w_2, s_1\}$ induces a $K_{1,4}$.

Therefore, $s_1x_1 \in E(G)$ and similarly, $s_1x_3 \in E(G)$, otherwise $N^I(w_3) \cup \{w_3, s_1\}$ induces a $K_{1,4}$. It follows that $\{s_1\} \cup N^I(s_1)$ induces a $K_{1,4}$. Similar argument holds for other vertices, and thus the vertices s_1, s_2, q_1, q_2 are adjacent to at least one of v_1, x_2 . Since the clique is maximal, there exists $v' \in K$ such that $v_1v' \notin E(G)$. We further classify based on the possibilities of v' as follows.

Case 1.1: $v' \in P_a$, i.e., w_1 or w_4 is non-adjacent to v_1 , say $v_1w_4 \notin E(G)$.

Note that either $v_2w_4 \in E(G)$ or $v_3w_4 \in E(G)$. Without loss of generality, let $v_2w_4 \in E(G)$. Note that v_3 is adjacent to at least one more clique vertex, say $v_3v'' \in E(G)$, where v'' is an end vertex of some path P_d . Further, we see the following cases depending on the possibilities of v'' .

Case 1.1.1: $v'' \notin P_a \cup P_b \cup P_c$. Then depending on the adjacency of s_1, s_2, q_1, q_2 to v_1, x_2 , one of the following is a desired path P .

If all of s_i, q_i are adjacent to v_1 , then $P = (\overrightarrow{P_d}v'', v_3, v, v_2, w_4 \overleftarrow{P_a}w_1, s_1, t_1, s_2, v_1, q_1, r_1, q_2)$

If all of s_i, q_i are adjacent to x_2 , then $P = (\overrightarrow{P_d}v'', v_3, v, v_2, w_4 \overleftarrow{P_a}w_3, v_1, w_2 \overleftarrow{P_a}w_1, s_1, t_1, s_2, x_2, q_1, r_1, q_2)$

If s_i, q_i are adjacent to different vertices in v_1, x_2 , say $v_1s_1, q_2x_2 \in E(G)$, then

$P = (\overrightarrow{P_d}v'', v_3, v, v_2, w_4 \overleftarrow{P_a}w_3, v_1, s_1, t_1, s_2, q_1, r_1, q_2, x_2, w_2 \overleftarrow{P_a}w_1).$

Case 1.1.2: $v'' \in P_a$.

Note that either $w_4x_1 \in E(G)$ or $w_4x_2 \in E(G)$, otherwise $N^I(w_2) \cup \{w_2, w_4\}$ has an induced $K_{1,4}$. Therefore, $v'' = w_1$, i.e., $v_3v'' \in E(G)$. Now we see the following sub cases. Let $S = \{s_1, s_2, q_1, q_2\}$. If at least three vertices in S are having adjacency with v_1 , say $s_1v_1, q_1v_1, q_2v_1 \in E(G)$. Note that all the vertices s_1, q_1, q_2 are having adjacency with one vertex in $N^I(w_3)$. We obtain desired path P as follows. If $q_2v_2 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, v_2, w_4 \overleftarrow{P_a}w_1, v_3, v).$

If $q_2x_3 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, x_3 \overleftarrow{P_a}w_1, v_3, v, v_2, w_3).$

If $q_2x_2 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, x_2 \overleftarrow{P_a}w_4, v_2, v, v_3, w_1, x_1, w_2).$

If $q_2x_1 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, x_1 \overleftarrow{P_a}w_4, v_2, v, v_3, w_1).$

Similarly, if at least three vertices in S are having adjacency with x_2 , say $s_1x_2, q_1x_2, q_2x_2 \in E(G)$. Note that all the vertices s_1, q_1, q_2 are having adjacency with one vertex in $\{v_2, v_3\}$. Further, we obtain P as follows. If $q_2v_2 \in E(G)$, then $P = (s_2, t_1, s_1, x_2, q_1, r_1, q_2, v_2, w_4, x_3, w_3, v_1, w_2, x_1, w_1, v_3, v).$

If $q_2v_3 \in E(G)$, then $P = (s_2, t_1, s_1, x_2, q_1, r_1, q_2, v_3, v, v_2, w_4 \overleftarrow{P_a}w_1).$

Now we see the case in which exactly two vertices in S are adjacent to both x_2, v_1 . If $s_1v_1, q_1v_1 \in E(G)$ and $s_2x_2, q_2x_2 \in E(G)$, then we obtain P as follows.

If $q_2v_2 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, v_2, w_4 \overleftarrow{P_a}w_1, v_3, v).$

If $q_2v_3 \in E(G)$, then $P = (s_2, t_1, s_1, v_1, q_1, r_1, q_2, v_3, v, v_2, w_4 \overleftarrow{P_a}w_1).$

If $s_1v_1, s_2v_1 \in E(G)$ and $q_1x_2, q_2x_2 \in E(G)$, then we obtain

$P = (s_2, t_1, s_1, v_1, w_2, x_1, w_1, v_3, v, v_2, w_4, x_3, w_2, x_2, q_1, r_1, q_2).$ The other cases are symmetric, when $s_1x_2, q_1x_2, s_2v_1, q_2v_1 \in E(G)$ and $q_1v_1, q_2v_1, s_1x_2, s_2x_2 \in E(G)$. This completes the Case 1.1.2.

Case 1.1.3: $v'' \in P_b \cup P_c$.

Without loss of generality, let $v'' = s_1$, i.e., $s_1v_3 \in E(G)$. Consider the adjacency of the vertices s_2, q_1 with v_1, x_2 . If $v_1s_2, v_1q_1 \in E(G)$, then the desired path $P = (q_2, r_1, q_1, v_1, s_2, t_1, s_1, v_3, v, v_2, w_4 \overleftarrow{P_a}w_1).$

If $x_2s_2, x_2q_1 \in E(G)$, then $P = (q_2, r_1, q_1, x_2, s_2, t_1, s_1, v_3, v, v_2, w_4 \overleftarrow{P_a}w_3, v_1, w_2 \overleftarrow{P_a}w_1).$ If $v_1s_2, x_2q_1 \in E(G)$, then $P = (w_1 \overleftarrow{P_a}w_3, v_1, s_2, t_1, s_1, v_3, v, v_2, w_4 \overleftarrow{P_a}x_2, q_1, r_1, q_2).$ The other case is symmetric when $x_2s_2, v_1q_1 \in E(G).$

Case 1.2: $v' \in \mathbb{P}_3$, without loss of generality $v_1s_1 \notin E(G).$

Note that $v_1w_i \in E(G)$, $1 \leq i \leq 4$. Note that either $s_1v_2 \in E(G)$ or $s_1v_3 \in E(G)$, say $s_1v_2 \in E(G)$. Since $s_1v_1 \notin E(G)$, clearly $s_1x_2 \in E(G)$. Since G is 2-connected, there exists $z' \in K$ such that $v_3z' \in E(G)$. Since there exists an adjacency for w_1, w_4 in $N^I(s_1)$, $d^I(w_i) = 3$, and thus $z' \neq w_i$, $1 \leq i \leq 4$. If $z' \notin P_a \cup P_b$, then z' is an end vertex of some path $P_d \in \mathbb{P}_3 \cup \mathbb{P}_1$, and $(\overrightarrow{P_d}z', v_3, v, v_2, s_1, t_1, s_2, v_1, w_1 \overleftarrow{P_a}w_4, \overrightarrow{P_c})$ or $(\overrightarrow{P_d}z', v_3, v, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, v_1, w_1, x_1, w_2, \overrightarrow{P_c})$ is the desired path.

If $z' \in P_c$, say $v_3q_1 \in E(G)$, then $(\overleftarrow{P_c}q_1, v_3, v, v_2, s_1, t_1, s_2, v_1, w_1 \overleftarrow{P_a}w_4)$ or

$(\overleftarrow{P_c}q_1, v_3, v, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, v_1, w_1, x_1, w_2)$ is the desired path.

Now the left over case is when $z' = s_2$. Now $d(v_3) = d(v_2) = d(t_1) = 2$, and $S = \{v_2, v_3, t_1\} \cup N(S)$

induces a short cycle. Since G has no short cycles, there exists some more adjacency for vertices in S . Observe that both w_1, w_4 have adjacency to a vertex in $N^I(s_1)$. Now if any one of w_1, w_4 is adjacent to any one of t_1, v_2 , then we obtain the desired path P as follows.

If $w_1v_2 \in E(G)$ and $q_1v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, w_4 \xrightarrow{\vec{P}_a} w_1, v_2, v, v_3, s_2, t_1, s_1)$.

If $w_1v_2 \in E(G)$ and $q_1x_2 \in E(G)$, then $P = (q_2, r_1, q_1, x_2 \xrightarrow{\vec{P}_a} w_4, v_1, w_2, x_1, w_1, v_2, v, v_3, s_2, t_1, s_1)$.

If $w_4v_2 \in E(G)$ and $q_1v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, w_1 \xrightarrow{\vec{P}_a} w_4, v_2, v, v_3, s_2, t_1, s_1)$.

If $w_4v_2 \in E(G)$ and $q_1x_2 \in E(G)$, then $P = (q_2, r_1, q_1, x_2 \xrightarrow{\vec{P}_a} w_1, v_1, w_3, x_3, w_4, v_2, v, v_3, s_2, t_1, s_1)$.

If $w_1t_1 \in E(G)$ and $q_1v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, w_4 \xrightarrow{\vec{P}_a} w_1, t_1, s_2, v_3, v, v_2, s_1)$.

If $w_1t_1 \in E(G)$ and $q_1x_2 \in E(G)$, then $P = (q_2, r_1, q_1, x_2 \xrightarrow{\vec{P}_a} w_4, v_1, w_2, x_1, w_1, t_1, s_2, v_3, v, v_2, s_1)$.

If $w_4t_1 \in E(G)$ and $q_1v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, w_1 \xrightarrow{\vec{P}_a} w_4, t_1, s_2, v_3, v, v_2, s_1)$.

If $w_4t_1 \in E(G)$ and $q_1x_2 \in E(G)$, then $P = (q_2, r_1, q_1, x_2 \xrightarrow{\vec{P}_a} w_1, v_1, w_3, x_3, w_4, t_1, s_2, v_3, v, v_2, s_1)$.

Now we shall consider the case where $w_1x_2, w_4x_2 \in E(G)$. Note that either $q_1v_1 \in E(G)$ or $q_1v_2 \in E(G)$ or $q_1v_3 \in E(G)$. If one of q_1, q_2 is adjacent to one of v_2, v_3 , then we obtain P as follows. If $q_1v_2 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1t_1, s_2, v_3, v, v_1, w_1 \xrightarrow{\vec{P}_a} w_4)$.

If $q_1v_3 \in E(G)$, then $P = (q_2, r_1, q_1, v_3, s_2t_1, s_1, v_2, v, v_1, w_1 \xrightarrow{\vec{P}_a} w_4)$.

Now if $q_i v_1 \in E(G)$, $i \in \{1, 2\}$, then q_i has an adjacency in $N^I(s_1)$. Further, if $q_1t_1 \in E(G)$, then $P = (q_2, r_1, q_1, t_1, s_2, v_3, v, v_2, s_1, x_2 \xrightarrow{\vec{P}_a} w_4, v_1, w_2, x_1, w_1)$.

Finally, we are left with one case that $q_ix_2 \in E(G)$. We claim that such a case cannot occur. Suppose not, then observe that $d^I(w_i) = d^I(s_j) = d^I(q_j) = 3$, $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$, and $d(v_2) = d(v_3) = d(t_1) = 2$. Clearly, $S = \{v_2, v_3, t_1\} \cup N(S)$ has a short cycle, a contradiction.

Case 1.3: $v' \notin P_a \cup \mathbb{P}_3$.

In this case we shall assume that $v_1w_i, v_1s_j, v_1q_j \in E(G)$, $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$. Note that $v'v_2 \in E(G)$ or $v'v_3 \in E(G)$. Assume without loss of generality that $v'v_2 \in E(G)$. Observe that there exists $w'' \in K$ such that w'' is an end vertex of some path P_d and $w''v_3 \in E(G)$. If $P_d \neq P_a$, $P_d \neq P_b$, and $P_d \neq P_c$, then $(\vec{P}_d w'', v_3, v, v_2, v', \vec{P}_a, v_1, \vec{P}_b, \vec{P}_c)$ is a desired path. On the other hand if P_d is one among P_a, P_b, P_c , say $P_d = P_b$, then $(\vec{P}_b w'', v_3, v, v_2, v', \vec{P}_a, v_1, \vec{P}_c)$ is a desired path.

Case 2: w_2, w_3 are adjacent to two different vertices in $N^I(v)$; i.e., without loss of generality, there exists $v_1, v_2 \in N^I(v)$ such that $v_1w_2, v_2w_3 \in E(G)$. Let $S = \{s_1, s_2, q_1, q_2\}$. Since G is 2-connected, observe that v_3 is adjacent to at least one more vertex w' in K . We observe the following possibilities.

Case 2.1: $w' \in P_a$. Without loss of generality let $w_1v_3 \in E(G)$. Now note that w_1 is adjacent to a vertex in $N^I(w_3)$. Thus we see the following sub cases.

Case 2.1.1: $w_1v_2 \in E(G)$.

Observe that the vertices in S are adjacent to any one of the following four vertex pairs; $\{v_2, x_1\}$, $\{v_2, x_2\}$, $\{v_2, v_1\}$, $\{v_3, x_2\}$.

Case 2.1.1.1: If there exists vertices in S having adjacency to v_2 . Without loss of generality, $s_1v_2, q_1v_2 \in E(G)$. Clearly, s_2, q_2 are adjacent to one of x_1, x_2, v_1 . We shall see the following arguments with respect to s_2 .

If $s_2x_1 \in E(G)$, then $P_1 = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_1, w_1, v_3, v, v_1, w_2, x_2, w_3, x_3, w_4)$ is a desired path in G . If $s_2v_1 \in E(G)$, then $P_2 = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, v_1, v, v_3, w_1, x_1, w_2, x_2, w_3, x_3, w_4)$ is a desired path in G . Now If $s_2x_2 \in E(G)$, then we see two more possibilities. If $s_2v_3 \in E(G)$. Note that in this case, s_1, q_1 are adjacent to x_2 , otherwise $N^I(q_1) \cup \{q_1, s_2\}$ induces a $K_{1,4}$. In this case $P_3 = (q_2, r_1, q_1, x_2, s_1, t_1, s_2, v_3, v, v_1, w_2, x_1, w_1, v_2, w_3, x_3, w_4)$ is a desired path in G .

The final case remaining is when all the vertices in S are adjacent to v_2 ; i.e., $s_1v_2, s_2v_2, q_1v_2, q_2v_2 \in E(G)$. Further, observe that if any of the vertex in S are adjacent to x_1 or v_1 , then we could obtain a similar path as that of P_1, P_2 , respectively. Therefore, we shall see the case when all the vertices of S are adjacent to v_2, x_2 . Now, note that $d^I(v_3) = d^I(v_1) = d^I(x_1) = 2$. Since G has no short cycles, there exists a vertex $w'' \in K$ such that w'' is adjacent to some vertices in $\{v_3, v_1, x_1\}$. If $w'' = w_4$, then we obtain the following desired paths.

If $w_4v_3 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, v_3, v, v_1, w_2, x_1, w_1)$.

If $w_4v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, v_1, w_2, x_1, w_1, v_3, v)$.

If $w_4x_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, x_1, w_1, v_3, v, v_1, w_2)$.

If w'' is an end vertex of a path P_d , then we obtain the desired path as follows.

If $w''v_3 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, \overrightarrow{P_d}w'', v_3, v, v_1, w_2, x_1, w_1)$.

If $w''v_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, \overrightarrow{P_d}w'', v_1, w_2, x_1, w_1, v_3, v)$.

If $w''x_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_2, w_3, x_3, w_4, \overrightarrow{P_d}w'', x_1, w_1, v_3, v, v_1, w_2)$.

The above cases hold true with respect to the vertex q_2 ; i.e., when q_2 is adjacent to one of x_1, x_2, v_1 .

Case 2.1.1.2: If all the vertices in S are non-adjacent to v_2 , then every vertices in S are adjacent to the vertices v_3, x_2 . In this case we obtain the same desired path P_3 as mentioned in the previous case.

Case 2.1.2: $w_1x_2 \in E(G)$

Similar to Case 2.1.1.

Case 2.1.3: $w_1x_3 \in E(G)$

Observe that all the vertices in S are adjacent to any one of the following three vertex pairs; $\{v_1, x_3\}$, $\{v_2, x_1\}$, $\{v_3, x_2\}$. If all the vertices in S are adjacent to $\{v_1, x_3\}$, then note that for every $u \in U = \{v_2, v_3, x_1, x_2\}$, $d^I(u) = 2$. Since G has no short cycles, there exists a vertex $w'' \in K$, such that for some $u \in U$, $w''u \in E(G)$. If $w'' = w_4$, then we obtain the following desired paths in G .

If $w_4v_3 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, v_3, w_1, x_1, w_2, x_2, w_3, v_2, v)$.

If $w_4v_2 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, v_2, v, v_3, w_1, x_1, w_2, x_2, w_3)$.

If $w_4x_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, x_1, w_2, x_2, w_3, v_2, v, v_3, w_1)$.

If $w_4x_2 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, x_2, w_3, v_2, v, v_3, w_1, x_1, w_2)$.

If w'' is an end vertex of a path P_d , then we obtain the desired path as follows.

If $w''v_3 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, \overrightarrow{P_d}w'', v_3, w_1, x_1, w_2, x_2, w_3, v_2, v)$.

If $w''v_2 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, \overrightarrow{P_d}w'', v_2, v, v_3, w_1, x_1, w_2, x_2, w_3)$.

If $w''x_1 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, \overrightarrow{P_d}w'', x_1, w_2, x_2, w_3, v_2, v, v_3, w_1)$.

If $w''x_2 \in E(G)$, then $P = (q_2, r_1, q_1, v_1, s_1, t_1, s_2, x_3, w_4, \overrightarrow{P_d}w'', x_2, w_3, v_2, v, v_3, w_1, x_1, w_2)$.

If all the vertices in S are adjacent to $\{v_2, x_1\}$, then $(q_2, r_1, q_1, v_2, s_1, t_1, s_2, x_1, w_1, v_3, v, v_1, w_2, x_2, w_3, x_3, w_4)$ is a desired path in G . If all the vertices in S are adjacent to $\{v_3, x_2\}$, then $(q_2, r_1, q_1, x_2, s_1, t_1, s_2, v_3, w_1, x_1, w_2, v_1, v, v_2, w_3, x_3, w_4)$ is a desired path in G .

Case 2.2: $w' \notin P_a$. In this case we shall assume that w_1, w_4 are adjacent to v_1 or v_2 . Without loss of generality, let $w' \in P_b$; i.e., $w' = s_1, s_1v_3 \in E(G)$. Observe that all the vertices in S are adjacent to any one of the following six vertex pairs; $\{v_1, x_2\}$, $\{v_1, x_3\}$, $\{v_2, v_1\}$, $\{v_2, x_1\}$, $\{v_2, x_2\}$, $\{v_3, x_2\}$. Clearly, $s_1v_3, s_1x_2 \in E(G)$. Note that all the vertices q_1, q_2 are adjacent to x_2 , otherwise suppose q_1 is not adjacent to the vertices x_2 , then $N^I(s_1) \cup \{s_1, q_1\}$ induces a $K_{1,4}$. Similar argument holds for q_2 . Moreover, q_1 is adjacent to a vertex in v_1, v_2, v_3 . If $w_1v_1, w_4v_1 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_2, w_3, x_3, w_4, v_1, w_1, x_1, w_2, x_2, q_1, r_1, q_2)$ is a desired path in G . If $w_1v_2, w_4v_2 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_1, w_2, x_1, w_1, v_2, w_4, x_3, w_3, x_2, q_1, r_1, q_2)$ is a desired path in G . If $w_1v_1, w_4v_2 \in E(G)$, then we see the adjacency of q_1 .

If $q_1v_1 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_2, w_4, x_3, w_3, x_2, q_2, r_1, q_1, v_1, w_2, x_1, w_1)$ is a desired path in G .

If $q_1v_2 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_1, w_1, x_1, w_2, x_2, q_2, r_1, q_1, v_2, w_3, x_3, w_4)$ is a desired path in G .

If $q_1v_3 \in E(G)$, then $(s_2, t_1, s_1, v_3, q_1, r_1, q_2, x_2, w_3, x_3, w_4, v_2, v, v_1, w_1, x_1, w_2)$ is a desired path in G .

If $w_1v_2, w_4v_1 \in E(G)$, then we see the adjacency of q_1 .

If $q_1v_1 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_2, w_1, x_1, w_2, v_1, q_1, r_1, q_2, x_2, w_3, x_3, w_4)$ is a desired path in G .

If $q_1v_2 \in E(G)$, then $(s_2, t_1, s_1, v_3, v, v_1, w_2, x_1, w_1, v_2, q_1, r_1, q_2, x_2, w_3, x_3, w_4)$ is a desired path in G .

If $q_1v_3 \in E(G)$, then $(s_2, t_1, s_1, v_3, q_1, r_1, q_2, x_2, w_3, x_3, w_4, v_1, v, v_2, w_1, x_1, w_2)$ is a desired path in G .

If $w' \in P_c$ or $w' \notin S$, then similar argument could be made and the desired path is obtained.

This completes the case analysis and a proof of the claim. \square

Claim 13 If $\mathbb{P}_j = \emptyset, j \geq 7$, and $\mathbb{P}_5 \neq \emptyset$, then $|\mathbb{P}_5| \leq 2$.

Proof. For a contradiction assume that there exists paths $P_a, P_b, P_c \in \mathbb{P}_5$ such that $P_a = (w_1, w_2, w_3; x_1, x_2)$, $P_b = (s_1, s_2, s_3; t_1, t_2)$, and $P_c = (q_1, q_2, q_3; r_1, r_2)$. From Claim 3, there exists $v_1 \in N^I(v)$ such that $v_1w_2, v_1s_2, v_1q_2 \in E(G)$. Since the clique K is maximal, there exists $w' \in K$ such that $w'v_1 \notin E(G)$.

Therefore, w' is an end vertex of some path in $\mathbb{P}_i, i \in \{1, 3, 5\}$. We see the following cases.

Case 1: w' is an end vertex of a P_5 .

Without loss of generality assume $w' = w_1$, i.e., $w_1v_1 \notin E(G)$. Note that $w_1v_2 \in E(G)$ or $w_1v_3 \in E(G)$. Observe that either $w_1t_1 \in E(G)$ or $w_1t_2 \in E(G)$, otherwise, $N^I(s_2) \cup \{s_2, w_1\}$ induces a $K_{1,4}$. Further, either $w_1r_1 \in E(G)$ or $w_1r_2 \in E(G)$, otherwise, $N^I(q_2) \cup \{q_2, w_1\}$ induces a $K_{1,4}$. Clearly, $\{w_1\} \cup N^I(w_1)$ induces a $K_{1,4}$, and thus $w' \notin \mathbb{P}_5$.

Case 2: $w' \in \mathbb{P}_3 \cup \mathbb{P}_1$.

Note that $w'v_2 \in E(G)$ or $w'v_3 \in E(G)$. Observe that either $w'x_1 \in E(G)$ or $w'x_2 \in E(G)$, otherwise, $N^I(w_2) \cup \{w_2, w'\}$ induces a $K_{1,4}$. Similarly, either $w't_1 \in E(G)$ or $w't_2 \in E(G)$, otherwise, $N^I(s_2) \cup \{s_2, w'\}$ induces a $K_{1,4}$. Further, either $w'r_1 \in E(G)$ or $w'r_2 \in E(G)$, otherwise, $N^I(q_2) \cup \{q_2, w'\}$ induces a $K_{1,4}$. It follows that, $\{w'\} \cup N^I(w')$ has an induced $K_{1,4}$. This contradicts the assumption that there exists three such paths P_a, P_b, P_c . This completes the cases analysis and a proof. \square

Claim 14 *If there exists $P_a, P_b \in \mathbb{P}_5, P_c \in \mathbb{P}_3$ and G has no short cycles, then G has a Hamiltonian cycle.*

Proof. Let $P_a, P_b \in \mathbb{P}_5, P_c \in \mathbb{P}_3$ such that $P_a = (w_1, w_2, w_3; x_1, x_2)$, $P_b = (s_1, s_2, s_3; t_1, t_2)$, and $P_c = (q_1, q_2; r_1)$. From Claim 13, it follows that there are no more 5-vertex paths in \mathbb{C} other than P_a, P_b . By Claim 3, there exists $v_1 \in N^I(v)$ such that $v_1w_2, v_1s_2 \in E(G)$. Now we claim that $v_1q_1, v_1q_2 \in E(G)$. Suppose if $v_1q_1 \notin E(G)$, then either q_1v_2 or $q_1v_3 \in E(G)$. Further, $q_1x_1 \in E(G)$ or $q_1x_2 \in E(G)$, otherwise $N^I(w_2) \cup \{w_2, q_1\}$ has an induced $K_{1,4}$. Either $q_1t_1 \in E(G)$ or $q_1t_2 \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, q_1\}$ has an induced $K_{1,4}$. It follows that $\{q_1\} \cup N^I(q_1)$ induces a $K_{1,4}$, a contradiction to the assumption that $v_1q_1 \notin E(G)$. Similarly, it is easy to see that $v_1q_2 \in E(G)$. Since the clique is maximal, there exists a vertex $w' \in K$ such that $w'v_1 \notin E(G)$. We observe the following cases depending on the choice of w' .

Case 1: $w' \in P_a \cup P_b$.

Without loss of generality let $v_1w_3 \notin E(G)$. From Claim A, either $w_3v_2 \in E(G)$ or $w_3v_3 \in E(G)$. Further, let us assume without loss of generality $w_3v_2 \in E(G)$. Now, observe that $w_3t_1 \in E(G)$ or $w_3t_2 \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, w_3\}$ induces a $K_{1,4}$. Note that either $w_3t_1 \in E(G)$ or $w_3t_2 \in E(G)$, otherwise $N^I(s_2) \cup \{s_2, w_3\}$ induces a $K_{1,4}$. Without loss of generality, we shall assume that $w_3t_2 \in E(G)$. We see the following cases.

Case 1.1: $v_1w_1 \in E(G)$.

Note that w_1 is adjacent to one of the vertices in $N^I(w_3)$, otherwise $N^I(w_3) \cup \{w_3, w_1\}$ induces a $K_{1,4}$. Observe that $q_i, i \in \{1, 2\}$ is adjacent to one of the vertices in $N^I(w_3)$. Thus $d^I(w_i) = d^I(q_j)3, i \in \{1, 2, 3\}, j = \{1, 2\}$. Now, if there exists $w'' \in K, w''v_3 \in E(G)$ such that $w'' \notin K' = \{w_1, w_2, w_3, s_1, s_2, s_3, q_1, q_2\}$, then w'' is an end vertex of some path $P_d \in \mathbb{P}_3 \cup \mathbb{P}_1$, then we obtain the following path as a desired path. $\overrightarrow{P_d}w'', v_3, v, v_2, w_3\overrightarrow{P_a}w_1, v_1, \overrightarrow{P_c}, \overrightarrow{P_b}$. If $w'' \in K'$, then either $w'' = s_1$ or $w'' = s_2$. If $w'' = s_1$, then observe that $(s_3\overrightarrow{P_b}s_1, v_3, v, v_2, w_3\overrightarrow{P_a}w_1, v_1, \overrightarrow{P_c})$ is a desired path.

Case 1.2: $v_1w_1 \notin E(G)$.

Note that $d^I(w_2) = d^I(w_3) = d^I(s_2) = 3$, and note that w_1 is adjacent to one of v_2, v_3 and also adjacent to one of t_1, t_2 . Further, if $w_1v_3 \in E(G)$, then $w_1t_2 \in E(G)$. That is, either $w_1v_3, w_1t_2 \in E(G)$ or $w_1v_2, w_1t_2 \in E(G)$ or $w_1v_2, w_1t_1 \in E(G)$. We detail the cases as follows.

Case 1.2.1: $w_1v_3, w_1t_2 \in E(G)$.

Note that $d^I(v_3) = d^I(v_2) = d^I(x_1) = d^I(x_2) = 2$ and $S = \{v_2, v_3, x_1, x_2\} \cup N(S)$ has a short cycle. Since G has no short cycles, it follows that there exists $w'' \in K$ such that for some $x \in S, w''x \in E(G)$. If w'' is an end vertex of a path $P_d \neq P_a \neq P_b \neq P_c$, then one of the following is a desired path P .

If $w''v_3 \in E(G)$, then $P = (\overrightarrow{P_d}w'', v_3, w_1\overrightarrow{P_a}w_3, v_2, v, v_1, \overrightarrow{P_c}, \overrightarrow{P_b})$

If $w''v_2 \in E(G)$, then $P = (\overrightarrow{P_d}w'', v_2, w_3\overrightarrow{P_a}w_1, v_3, v, v_1, \overrightarrow{P_c}, \overrightarrow{P_b})$

If $w''x_1 \in E(G)$, then $P = (\overrightarrow{P_d}w'', x_1, w_1, v_3, v, v_2, w_3, x_2, w_2, v_1, \overrightarrow{P_c}, \overrightarrow{P_b})$

If $w''x_2 \in E(G)$, then $P = (\overrightarrow{P_d}w'', x_2, w_3, v_2, v, v_3, w_1, x_1, w_2, v_1, \overrightarrow{P_c}, \overrightarrow{P_b})$

If $w'' \in P_b$, then the path P_d could be replaced by the path P_b and remove the last occurrence of P_b to get the desired path in all of the above cases.

Case 1.2.2: $w_1v_2, w_1t_2 \in E(G)$ or $w_1v_2, w_1t_1 \in E(G)$.

Since G is 2-connected, there exists a vertex $w'' \in K$ such that w'' is an end vertex of a path $P_d \neq P_a \neq P_b \neq P_c$ such that $w''v_3 \in E(G)$, then $(\vec{P_d}w'', v, v_2, w_1\vec{P_a}w_3, t_2, s_3, s_1, t_1, s_2, v_1, \vec{P_c})$ is a desired path. Note that q_1, q_2 are adjacent to a vertex in $N^I(w_3)$. Therefore $w'' \notin P_c$. If $w'' \in P_b$, then we obtain the following observations. If $s_1v_3 \in E(G)$ and $s_3v_3 \in E(G)$, then $(\vec{P_c}, v_1, s_2, t_1, s_1, v_3, s_3, t_2, w_3\vec{P_a}w_1, v_2, v)$ is a desired path. If $s_1v_3 \in E(G)$ and $s_3v_3 \notin E(G)$, then $(\vec{P_c}, v_1, s_2, t_1, s_1, v_3, v, v_2, w_1\vec{P_a}w_3, t_2, s_3)$. Finally we shall see the case that $s_1v_3 \notin E(G)$ and $s_3v_3 \in E(G)$. In this case, note that either $s_1v_1 \in E(G)$ or $s_1v_2 \in E(G)$. If $s_1v_1 \in E(G)$, then $(\vec{P_c}, v_1, s_1\vec{P_b}s_3, v_3, v, v_2, w_3\vec{P_a}w_1)$ is a desired path. If $s_1v_2 \in E(G)$, then $(w_1\vec{P_a}w_3, v_2, s_1\vec{P_b}s_3, v_3, v, \vec{P_c})$ is a desired path.

Case 2: $w' \notin P_a \cup P_b$.

In this case we assume that $v_1w \in E(G)$, $w \in \{w_i, s_i, q_j\}, i \in \{1, 2, 3\}, j \in \{1, 2\}$. Clearly $w'v_2 \in E(G)$ or $w'v_3 \in E(G)$. Without loss of generality assume that $w'v_2 \in E(G)$. We have already shown that $w' \notin P_c$. Thus w' is an end vertex of a path P_e . Since G is 2-connected, v_3 is adjacent to end vertex of a path P_f . Note that $P_f \neq P_e$, otherwise $C = (v_3, \vec{P_e}, v_2, v, v_3)$ is a short cycle in G . Thus $P = (\vec{P_e}, v_2, v, v_3, \vec{P_f}, \vec{P_a}, v_1, \vec{P_b}, \vec{P_c})$ is a desired path. Note that if P_f is some paths among P_a, P_b, P_c , say $P_f = P_a$, then the desired path is $(\vec{P_e}, v_2, v, v_3, \vec{P_f}, \vec{P_b}, v_1, \vec{P_c})$. Similarly we could easily obtain if $P_f = P_b$ and $P_f = P_c$.

This completes the case analysis and a proof. \square

Observation: In the following claims, we will be producing a cycle C containing all the vertices of $N^I(v)$ and a path P_m not in C . In this situation, it is easy to see that P_m is adjacent to at least one vertex in $\{v_1, v_2, v_3\}$, say v_1 . Further, $\vec{P_m}, v_1 \vec{C} v_1^-$ is a desired path in G , where the vertices v_1^-, v_1 occur consecutively in \vec{C} .

Claim 15 Let $P_a, P_b, P_c \in \mathbb{P}_3$, and $v_1 \in N^I(v)$ such that v_1 is adjacent to end vertex of at least two paths in P_a, P_b, P_c , $|\mathbb{P}_5| \leq 1$, and $\mathbb{P}_j = \emptyset, j \geq 7$. If G has no short cycles, then G has a Hamiltonian cycle.

Proof. Let $P_a = (w_1, w_2; x_1)$, $P_b = (s_1, s_2; t_1)$, $P_c = (q_1, q_2; r_1)$. Without loss of generality, assume that $w_2v_1, s_1v_1 \in E(G)$. Since the clique is maximal, there exists $w' \in K$ such that $w'v_1 \notin E(G)$. We see the following cases.

Case 1: w' is one among w_1, s_2 .

Without loss of generality, let us assume that $v_1w_1 \notin E(G)$. Note that either $w_1v_2 \in E(G)$ or $w_1v_3 \in E(G)$. Let us assume without loss of generality that $w_1v_2 \in E(G)$. Since G is 2-connected, v_3 is adjacent to at least one more vertex $w'' \in K$. Depending on the possibilities for w'' we see the following cases.

Case 1.1 $w'' = w_1$. Note that q_1 is adjacent to one of the vertices in $\{v_2, v_3, x_1\}$.

If $q_1x_1 \in E(G)$, then $P = (\vec{P_c}q_1, x_1, w_1, v_2, v, v_3, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

If $q_1v_2 \in E(G)$, then $P = (\vec{P_c}q_1, v_2, w_1, x_1, w_2, v_3, v, v_1, s_1, t_1, s_2)$ is a desired path.

If $q_1v_3 \in E(G)$, then $P = (\vec{P_c}q_1, v_3, v, v_2, w_1, x_1, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

Case 1.2 $w'' = w_2$. Note that $d^I(v_3) = d^I(v_2) = d^I(x_1) = 2$. Since G has no short cycles, at least one vertex of v_3, v_2, x_1 has at least one more adjacency in K . If $w_1v_3 \in E(G)$, then by Case 1.1, there exists a desired path. In all further possibilities, we produces a cycle C containing all the vertices in $N^I(v)$, and path P_c not in C . Thus by previous observation, there exists a desired path in G .

If $s_2v_3 \in E(G)$, then $C = (s_2, t_1, s_1, v_1, w_2, x_1, w_1, v_2, v, v_3, s_2)$.

If $s_2v_2 \in E(G)$, then $C = (s_2, t_1, s_1, v_1, v, v_3, w_2, x_1, w_1, v_2, s_2)$.

If $s_2x_1 \in E(G)$, then $C = (s_2, t_1, s_1, v_1, w_2, v_3, v, v_2, w_1, x_1, s_2)$.

If s_1 is adjacent to v_3, v_2, x_1 , then we shall consider the adjacency of s_2 also. That is, s_2 is adjacent to one of v_1, v_2, v_3 . Since we have seen the case when $s_2v_3 \in E(G)$, and $s_2v_2 \in E(G)$, the remaining case is when $s_2v_1 \in E(G)$.

If $s_2v_1, s_1v_3 \in E(G)$, then $C = (s_2, t_1, s_1, v_3, v, v_2, w_1, x_1, w_2, v_1, s_2)$.

If $s_2v_1, s_1v_2 \in E(G)$, then $C = (s_2, t_1, s_1, v_2, w_1, x_1, w_2, v_3, v, v_1, s_2)$.

If $s_2v_1, s_1x_1 \in E(G)$, then $C = (s_2, t_1, s_1, x_1, w_1, v_2, v, v_3, w_2, v_1, s_2)$.

If there exists a vertex $w^* \notin \{w_1, s_1, s_2\}$ adjacent to vertices in $\{v_3, v_2, x_1\}$, then note that w^* is an end vertex of some path P_d . We get desired path as follows.

If $w^*x_1 \in E(G)$, then $P = (\overrightarrow{P_d}w^*, x_1, w_1, v_2, v, v_3, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

If $w^*v_2 \in E(G)$, then $P = (\overrightarrow{P_d}w^*, v_2, w_1, x_1, w_2, v_3, v, v_1, s_1, t_1, s_2)$ is a desired path.

If $w^*v_3 \in E(G)$, then $P = (\overrightarrow{P_d}w^*, v_3, v, v_2, w_1, x_1, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

Case 1.3 $w'' = s_1$. Here $N^I(s_1) = \{v_1, t_1, v_3\}$. Clearly, q_1 is adjacent to one of the vertices in $\{v_1, v_2, v_3\}$.

If $q_1v_1 \in E(G)$, then $P = (\overrightarrow{P_c}q_1, v_1, w_2, x_1, w_1, v_2, v, v_3, s_1, t_1, s_2)$ is a desired path.

If $q_1v_2 \in E(G)$, then $P = (\overrightarrow{P_c}q_1, v_2, w_1, x_1, w_2, v_1, v, v_3, s_1, t_1, s_2)$ is a desired path.

If $q_1v_3 \in E(G)$, then $P = (\overrightarrow{P_c}q_1, v_3, v, v_2, w_1, x_1, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

Case 1.4 $w'' = s_2$.

In this case, note that the path P_c is not in the cycle $C = (s_2, t_1, s_1, v_1, w_2, x_1, w_1, v_2, v, v_3, s_2)$, and by previous Observation, we could easily get a desired path in G .

Case 1.5 $w'' \notin \{w_1, w_2, s_1, s_2\}$. Clearly w'' is an end vertex of some path P_d , and we obtain $P = (\overrightarrow{P_d}w'', v_3, v, v_2, w_1, x_1, w_2, v_1, s_1, t_1, s_2)$ is a desired path.

Case 2: w' is in $K \setminus \{w_1, s_2\}$.

Let $P_e = (y_1, y_2, y_3; z_1, z_2)$. In this case note that for every $u \in W = \{w_1, w_2, s_1, s_2\}$, $v_1u \in E(G)$. Since the clique is maximal, there exists $w^* \in K$ such that $v_1w^* \notin E(G)$, and w^* is adjacent to one of v_2, v_3 . Without loss of generality, let $w^*v_2 \in E(G)$. Now v_3 is adjacent to at least one more vertex $s^* \in K$. We see the following cases.

Case 2.1: w^* and s^* are end vertices of two different paths, say P_n, P_m , respectively.

In this case, if $P_m \neq P_a$ and $P_m \neq P_b$, then $\overrightarrow{P_a}, v_1, \overrightarrow{P_b}, \overrightarrow{P_n}w^*, v_2, v, v_3, s^*, \overleftarrow{P_m}$ is a desired path. If P_m is one among P_a, P_b say $v_3w_1 \in E(G)$, then $\overrightarrow{P_b}, v_1, w_2, x_1, w_1, v_3, v, v_2, w^*, \overleftarrow{P_n}$ is a desired path.

Case 2.2: w^* and s^* are end vertices of the same path, say P_n .

Case 2.2.1: $P_n \in \mathbb{P}_3$, say $P_n = P_c$, i.e., $q_1v_3, q_2v_2 \in E(G)$.

Now it is easy to see that $d^I(v_2) = d^I(v_3) = d^I(r_1) = 2$. Since G has no short cycles there exists $q^* \in K$ such that q^* is adjacent to at least one of v_2, v_3, r_1 . If $q^* \in \{w_1, w_2, s_1, s_2\}$, then we obtain the following desired paths. Without loss of generality let $q^* = w_1$.

If $w_1r_1 \in E(G)$, then $P_1 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, r_1, q_2, v_2, v, v_3, q_1)$.

If $w_1v_2 \in E(G)$, then $P_2 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, v_2, v, v_3, q_1, r_1, q_2)$.

If $w_1v_3 \in E(G)$, then $P_3 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, v_3, v, v_2, q_2, r_1, q_1)$. Note that if $q^* \in \{q_1, q_2\}$, say q_1 , then $q_1v_2 \in E(G)$. Further, $d^I(q_1) = 3$, and for every $u \in \{w_1, w_2, s_1, s_2\}$, u is adjacent to one of v_3, v_2, r_1 . It follows that the above paths P_1, P_2, P_3 will be the desired paths. A symmetric argument holds if $q^* = q_2$. Finally, if q^* is some vertex other than $w_1, w_2, s_1, s_2, q_1, q_2$, then observe that q^* is either an end vertex of some path P_d or $q^* = y_2$, the middle vertex of path P_e . If q^* is an end vertex of P_d , then we obtain the following desired paths.

If $q^*r_1 \in E(G)$, then $P_1 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, \overrightarrow{P_d}q^*, r_1, q_2, v_2, v, v_3, q_1)$.

If $q^*v_2 \in E(G)$, then $P_2 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, \overrightarrow{P_d}q^*, v_2, v, v_3, q_1, r_1, q_2)$.

If $q^*v_3 \in E(G)$, then $P_3 = (\overrightarrow{P_b}, v_1, w_2, x_1, w_1, \overrightarrow{P_d}q^*, v_3, v, v_2, q_2, r_1, q_1)$. Now, if $q^* = y_2$, then note that $d^I(y_2) = 3$. It follows that w_1 is adjacent to a vertex in $N^I(y_2)$. Since we have already seen the case when $q^* = w_1$, we shall see the remaining cases as follows.

If $w_1z_1, y_2r_1 \in E(G)$, then $P = (y_1, z_1, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_3, z_2, y_2, r_1, q_2, v_2, v, v_3, q_1)$.

If $w_1z_1, y_2v_2 \in E(G)$, then $P = (y_1, z_1, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_3, z_2, y_2, v_2, v, v_3, q_1, r_1, q_2)$.

If $w_1z_1, y_2v_3 \in E(G)$, then $P = (y_1, z_1, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_3, z_2, y_2, v_3, v, v_2, q_2, r_1, q_1)$.

If $w_1z_2, y_2r_1 \in E(G)$, then $P = (y_3, z_2, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_1, z_1, y_2, r_1, q_2, v_2, v, v_3, q_1)$.

If $w_1z_2, y_2v_2 \in E(G)$, then $P = (y_3, z_2, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_1, z_1, y_2, v_2, v, v_3, q_1, r_1, q_2)$.

If $w_1z_2, y_2v_3 \in E(G)$, then $P = (y_3, z_2, w_1, \overrightarrow{P_a}, v_1, \overrightarrow{P_b}, y_1, z_1, y_2, v_3, v, v_2, q_2, r_1, q_1)$.

Case 2.2.2: $P_n \in \mathbb{P}_5$, say $P_n = P_e$, i.e., $y_1v_3, y_3v_2 \in E(G)$.

Clearly, $d^I(v_2) = d^I(v_3) = d^I(z_1) = d^I(z_2) = 2$. Since G has no short cycles there exists $q^* \in K$ such that q^* is adjacent to at least one of v_2, v_3, z_1, z_2 . If $q^* \in \{w_1, w_2, s_1, s_2\}$, then we obtain the following desired paths. Without loss of generality let $q^* = w_1$.

If $w_1v_2 \in E(G)$, then $P_1 = (\vec{P}_b, v_1, w_2, x_1, w_1, v_2, v, v_3, y_1 \vec{P}_e)$.

If $w_1v_3 \in E(G)$, then $P_2 = (\vec{P}_b, v_1, w_2, x_1, w_1, v_3, v, v_2, y_3 \vec{P}_e)$.

If $w_1z_1 \in E(G)$, then $P_3 = (\vec{P}_b, v_1, w_2, x_1, w_1, z_1 \vec{P}_e y_3, v_2, v, v_3, y_1)$.

If $w_1z_2 \in E(G)$, then $P_4 = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, v_2, v, v_3, y_1 \vec{P}_e y_2)$.

If $q^* \in \{y_1, y_2, y_3\}$, say y_1 then note that $d^I(y_1) = 3$. Since $N^I(y_1) \cup \{y_1, w_1\}$ has no induced $K_{1,4}$, w_1 is adjacent to some vertices in v_2, v_3, z_1, z_2 , and we obtain the desired paths P_1, P_2, P_3, P_4 , same as above. If $q^* \notin \{w_1, w_2, s_1, s_2, y_1, y_2, y_3\}$, then we obtain the following desired paths. Note that q^* is an end vertex of some path $P_d \in \mathbb{P}_3$.

If $q^*v_2 \in E(G)$, then $P_1 = (\vec{P}_b, v_1, w_2, x_1, w_1, \vec{P}_d q^*, v_2, v, v_3, y_1 \vec{P}_e)$.

If $q^*v_3 \in E(G)$, then $P_2 = (\vec{P}_b, v_1, w_2, x_1, w_1, \vec{P}_d q^*, v_3, v, v_2, y_3 \vec{P}_e)$.

If $q^*z_1 \in E(G)$, then $P_3 = (\vec{P}_b, v_1, w_2, x_1, w_1, \vec{P}_d q^*, z_1 \vec{P}_e y_3, v_2, v, v_3, y_1)$.

If $q^*z_2 \in E(G)$, then $P_4 = (\vec{P}_b, v_1, w_2, x_1, w_1, \vec{P}_d q^*, z_2, y_3, v_2, v, v_3, y_1 \vec{P}_e y_2)$.

Case 2.3: one of w^* and s^* is adjacent to y_2 .

Without loss of generality let us assume $v_2y_2 \in E(G)$. Note that $d^I(y_2) = 3$. We now claim that for every $u \in \{w_1, w_2, s_1, s_2\}$, $uv_3 \notin E(G)$. Suppose there exists an adjacency for v_3 in $P_a \cup P_b$, say $v_3w_1 \in E(G)$, then $N^I(y_2) \cup \{y_2, w_1\}$ induces a $K_{1,4}$. Therefore, the vertex s^* is such that $s^* \notin \{w_1, w_2, s_1, s_2\}$, $q^*v_3 \in E(G)$. We have the following possibilities for s^* .

Case 2.3.1: $s^* \in \{y_1, y_3\}$.

If $v_3y_1 \in E(G)$, then we obtain the desired path as follows. Now $d^I(v_2) = d^I(v_3) = d^I(z_1) = 2$. Since G has no short cycles, there exists a vertex $q^* \in K$ such that q^* is adjacent to at least one of v_2, v_3, z_1 . If $q^* \in \{w_1, w_2, s_1, s_2\}$, then without loss of generality, let us assume that $q^* = w_1$. We already observed that $w_1v_3 \notin E(G)$. Thus if $w_1v_2 \in E(G)$, then $P_1 = (\vec{P}_b, v_1, w_2, x_1, w_1, v_2, v, v_3, y_1 \vec{P}_e y_3)$.

If w_1z_1 , then $P_2 = (\vec{P}_b, v_1, w_2, x_1, w_1, z_1, y_1, v_3, v, v_2, y_2, z_2, y_3)$.

If $q^* = y_1$, then $y_1v_2 \in E(G)$, $d^I(y_1) = 3$, and w_1 is adjacent to at least a vertex in $N^I(y_1)$. Note that $w_1v_3 \notin E(G)$, and for the other two possibilities, we obtain the desired paths P_1, P_2 same as above. If $q^* = y_3$. Note that all the vertices in $W = \{w_1, w_2, s_1, s_2\}$ are adjacent to a vertex in $N^I(y_2)$. Since the case in which vertices in W are adjacent to v_2, z_1 are already analysed, without loss of generality we shall assume that for every $u \in \{w_1, w_2, s_1, s_2\}$, $uz_2 \in E(G)$.

If $y_3v_3 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, v_3, v, v_2, y_2, z_1, y_1)$.

If $y_3v_2 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, v_2, v, v_3, y_1, z_1, y_2)$.

If $y_3z_1 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, z_1, y_2, v_2, v, v_3, y_1)$.

Now we shall consider $q^* \notin \{w_1, w_2, s_1, s_2, y_1, y_3\}$. Here also, similar to the previous argument we shall assume that for every $u \in \{w_1, w_2, s_1, s_2\}$, $uz_2 \in E(G)$. Note that q^* is an end vertex of some path $P_d \in \mathbb{P}_3$.

If $q^*v_3 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, \vec{P}_d q^*, v_3, v, v_2, y_2, z_1, y_1)$.

If $q^*v_2 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, \vec{P}_d q^*, v_2, v, v_3, y_1, z_1, y_2)$.

If $q^*z_1 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, \vec{P}_d q^*, z_1, y_2, v_2, v, v_3, y_1)$.

If $v_3y_3 \in E(G)$, then the case is symmetric.

Case 2.3.2: $s^* \notin \{y_1, y_3\}$.

Note that s^* is an end vertex of a path $P_f \in \mathbb{P}_3$. Note that w_1 is adjacent to at least a vertex in $N^I(y_2)$. We obtain the desired paths as follows.

If $w_1v_2 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, v_2, v, v_3, q^* \vec{P}_f, \vec{P}_e)$.

If $w_1z_1 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_1, y_1, y_3, z_2, y_2, v_2, v, v_3, q^* \vec{P}_f)$.

If $w_1z_2 \in E(G)$, then $P = (\vec{P}_b, v_1, w_2, x_1, w_1, z_2, y_3, y_1, z_1, y_2, v_2, v, v_3, q^* \vec{P}_f)$.

This completes the case analysis and the proof. \square

Claim 16 *If there exists $P_a \in \mathbb{P}_5, P_b, P_c, P_d \in \mathbb{P}_3$ and G has no short cycles, then G has a Hamiltonian cycle.*

Proof. Note that there are four paths and thus there exists a vertex in $N^I(v)$, adjacent to at least two different paths. Without loss of generality, let us assume that the end vertices of two different paths are adjacent to $v_1 \in N^I(v)$. If two paths, say $P_b, P_c \in \mathbb{P}_3$ are adjacent to v_1 , then by Claim 15, G has a Hamiltonian cycle. On the other hand, we shall assume that no such two paths exists in \mathbb{P}_3 . Therefore, we could assume that the end vertices of P_b, P_c, P_d are adjacent to v_1, v_2, v_3 , respectively. Now end vertices of P_a is adjacent to a vertex in $N^I(v)$, say v_1 . Observe that $P = (\vec{P}_a, v_1, \vec{P}_b, \vec{P}_c, v_2, v, v_3, \vec{P}_d)$ is a desired path in G . This completes the proof. \square

Claim 17 *If $\mathbb{P}_j = \emptyset, j \geq 5$ and there exists $P_a, P_b, P_c, P_d, P_e \in \mathbb{P}_3$ and G has no short cycles, then G has a Hamiltonian cycle.*

Proof. Note that there exists five paths and at least two those paths are adjacent to one of v_1, v_2, v_3 . Observe that the premise of Claim 15 is satisfied, and therefore, G has a Hamiltonian cycle. \square

Theorem 2. *Let G be a 2-connected, $K_{1,4}$ -free split graph with $|K| \geq |I| \geq 8$. G has a Hamiltonian cycle if and only if there are no induced short cycles in G . Further, finding such a cycle is polynomial-time solvable.*

Proof. Necessity is trivial. Sufficiency follows from the previous claims.

3 Hardness Result

Akiyama et al. [21] proved the NP-completeness of Hamiltonian cycle in planar bipartite graphs with maximum degree 3. Here we give a reduction from Hamiltonian cycle problem in planar bipartite graphs with maximum degree 3 to Hamiltonian cycle problem in $K_{1,5}$ -free split graph.

Theorem 3. *Hamiltonian cycle problem in $K_{1,5}$ -free split graph is NP-complete.*

Proof. For NP-hardness result, we present a deterministic polynomial-time reduction that reduces an instance of planar bipartite graph with maximum degree 3 to the corresponding instance in split graphs. Consider a planar bipartite graph G with maximum degree 3, and let A, B be the partitions of $V(G)$. We construct two graphs H_1, H_2 from G as follows.

$$V(H_1) = V(H_2) = V(G), E(H_i) = E(G) \cup E_i, i \in \{1, 2\} \text{ where } E_1 = \{uv : u, v \in A\}, E_2 = \{uv : u, v \in B\}$$

Clearly, the reduction is a polynomial-time reduction and H_1, H_2 are split graphs with maximal cliques A, B , respectively. We now show that there exists a Hamiltonian cycle in G if and only if there exists a Hamiltonian cycle in H_1 and there exists a Hamiltonian cycle in H_2 . Note that, if G has a Hamiltonian cycle, then $|A| = |B|$.

Necessity: If there exists a Hamiltonian cycle C in G , then C is a Hamiltonian cycle in H_1 and H_2 , since G is a strict subgraph of H_1 and similarly, that of H_2 .

Sufficiency: Since there exists a Hamiltonian cycle in H_1 , $|A| \geq |B|$ and since H_2 has a Hamiltonian cycle, $|A| \leq |B|$. It follows that $|A| = |B|$. Now we claim that any Hamiltonian cycle C in H_1 is also a Hamiltonian cycle in G . If not, there exists at least one edge $uv \in E(C)$ where $u, v \in A$. It follows that at least one vertex in B is not in C , which contradicts the Hamiltonicity of H_1 . Hence the sufficiency follows. We now show that the constructed graphs H_1 and H_2 are $K_{1,5}$ -free. Suppose there exists a $K_{1,5}$ in H_1 or H_2 induced on vertices $\{u, v, w, x, y, z\}$, centered at v . At most two vertices (say u, v) of $K_{1,5}$ belongs to the clique K . Therefore, $w, x, y, z \in I$ and this implies $d_{H_i}^I(v) \geq 4, i = 1, 2$. It follows that $d_G(v) = 4$, which is a contradiction to the maximum degree of the bipartite graph G . Since a given instance of Hamiltonian problem in $K_{1,5}$ -free split graphs can be verified in deterministic polynomial time, the problem is in class NP. It follows that the Hamiltonian cycle problem in $K_{1,5}$ -free split graphs is NP-complete and the theorem follows. \square

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